Packing nearly optimal Ramsey $R(3, t)$ graphs

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Joint work with Lutz Warnke

Context of this talk

Ramsey number $R(s,t)$

$R(s,t) :=$ minimum $n \in \mathbb{N}$ such that every red/blue edge-colouring of complete n-vertex graph K_n contains red K_s or blue K_t

- Major problem in combinatorics: determining asymptotics
- Testbed for new proof techniques/methods:
  Alteration, LLL, Concentration Ineq., Semi-Random, Differential Eq.

Celebrated Result (Ajtai-Komlós-Szemerédi 1980 + Kim 1995)

$R(3,t) = \Theta(t^2/\log t)$

- Lower bound harder: Kim received Fulkerson Prize 1997
- $R(3,t) = \Omega(t^2/(\log t)^2)$ already by Erdős in 1961

Topic of this talk

Extension of Kim-result (implies asymptotics of other Ramsey parameter)

MAIN RESULT: NEARLY OPTIMAL $R(3, t)$ GRAPHS

Kim (1995) + Bohman (2008): one nearly optimal $R(3, t)$ graph

Both find an n-vertex graph $G \subseteq K_n$ such that
G is Δ-free with independence number $\alpha(G) \leq C\sqrt{n \log n}$

- Using (semi-random variation of) Δ-*free process*:
  greedily add random edges that do not close a Δ

G., Warnke (2017+): almost packing of nearly optimal $R(3, t)$ graphs

Given $\varepsilon > 0$, we find edge-disjoint graphs $(G_i)_{i \in \mathcal{I}}$ with $G_i \subseteq K_n$ such that
(a) each G_i is Δ-free with $\alpha(G_i) \leq C_\varepsilon \sqrt{n \log n}$
(b) the union of the G_i contains $\geq (1 - \varepsilon)\binom{n}{2}$ edges

- Using simple *polynomial-time randomized algorithm*:
  sequentially choose G_i via semi-random variation of Δ-free process

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Motivation: why should we care?

- Natural packing extension of Kim's result
- Technical challenge: controlling errors over $\Theta(\sqrt{n/\log n})$ iterations
- Establishes Ramsey-Theory conjecture by Fox et.al. (cf. next slides)

Ramsey Theory with $r \geq 2$ colours

$G \to (H)_r$ $:\Leftrightarrow$ any r-colouring of $E(G)$ has monochromatic copy of H

Ramsey theory $\triangleq$ studying properties of 'r-Ramsey minimal graphs'

$\mathcal{M}_r(H) :=$ all graphs G that are r-Ramsey minimal for H
(i.e., $G \to (H)_r$ and $G' \not\to (H)_r$ for all $G' \subsetneq G$)

- $\min_{G \in \mathcal{M}_r(K_k)} v(G) =$ Ramsey number
- $\min_{G \in \mathcal{M}_r(K_k)} e(G) =$ Size Ramsey number

Minimum degree of r-Ramsey minimal graphs (Burr, Erdős, Lovász 1976)

$s_r(H) := \min_{G \in \mathcal{M}_r(H)} \delta(G)$

- $s_2(K_k) = (k-1)^2$: Burr, Erdős, Lovász (1976)
- $s_2(H) = 2\delta(H) - 1$: for many bipartite H (trees, $K_{a,b}$, etc)
  Fox, Lin (2006) + Szabó, Zumstein, Zürcher (2010)
- $s_r(K_k) = \tilde{\Theta}_k(r^2)$: Fox, Grinshpun, Liebenau, Person, Szabó (2015)

Ramsey Conjecture of Fox et.al.

Minimum degree of minimal r-Ramsey graphs (Burr, Erdős, Lovász 1976)

$s_r(K_k) := min_{G \in \mathcal{M}_r(K_k)} \delta(G)$

- $cr^2 \log r \le s_r(K_3) \le Cr^2(\log r)^2$ by FGLPS (2015)

Conjecture (Fox, Grinshpun, Liebenau, Person, Szabo, 2015)

$s_r(K_3) = O(r^2 \log r)$

- They suggested to pack G_i sequentially via Δ-free process
  (their weaker upper bound relies on sequential LLL–argument)

Conj. True (G., Warnke, 2017+): corollary of our main packing result

Implies $s_r(K_3) = \Theta(r^2 \log r)$

- For technical reasons: use *semi-random variation* of Δ-free process

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Glimpse of the proof

Main-Technical-Result: find random-like Δ-free subgraph $G \subseteq H$

Let $p := \sqrt{\delta(\log n)/n}$ and $s := C_{\varepsilon,\delta}\sqrt{n \log n}$. If $H \subseteq K_n$ is such that

$$e_H(A,B) \geq \varepsilon\,|A||B|$$

for all disjoint sets A, B of size s, then we can find Δ-free $G \subseteq H$ with

$$e_G(A,B) = (1 \pm \delta)\,p\,e_H(A,B)$$

for all disjoint A, B of size s.

Proof based on semi-random variation of Δ-free process:

- Do *not* require degree/codegree regularity of H
- 'Self-stabilization' mechanism built into process (to control errors)
- Tools: Bounded-Differences-Ineq. and Upper-Tail-Ineq. of Warnke

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Implies packing result: *(maintaining $e_{H_i}(A, B)$ bounds inductively)*

- Start with $H_0 = K_n$
- Sequentially choose $G_i \subseteq H_i$ and set $H_{i+1} = H_i \setminus G_i$
- Stop when $e_{H_I}(A, B) \approx \varepsilon |A||B|$ holds

Summary

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Given $\varepsilon > 0$, we find edge-disjoint graphs $(G_i)_{i \in \mathcal{I}}$ with $G_i \subseteq K_n$ such that
(a) each G_i is Δ-free with $\alpha(G_i) \le C_\varepsilon \sqrt{n \log n}$
(b) the union of the G_i contains $\ge (1-\varepsilon)\binom{n}{2}$ edges

Remarks

- Natural algorithmic packing version of Kim's $R(3,t)$ construction
- Establishes $s_r(K_3) = \Theta(r^2 \log r)$ asymptotics conjectured by Fox et.al.

Questions

- Further applications of the K_3-free packing result?
- Generalization of packing-result to K_k-free graphs worth effort?