Packing Nearly Optimal Ramsey $R(3,t)$ Graphs

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Abstract/Summary

- Previous work: Kim/Bohman **construct one nearly optimal** $R(3,t)$ **graph** via (semi-random) triangle-free process
- Our contribution: We **approximately decompose** K_n **into nearly optimal** $R(3,t)$ **graphs** via iterative semi-random Δ-free process
- Our application: We **solve Ramsey-Conjecture by Fox et al.** (determine order of Ramsey-type parameter of Burr-Erdős-Lovász from 1976)

Kim (1995), Bohman (2008): one nearly optimal $R(3,t)$ graph

They both find an n-vertex graph $G \subseteq K_n$ such that
G is Δ-free with independence number $\alpha(G) \le C\sqrt{n \log n}$

- Kim/Bohman use (semi-random) Δ-free process
- **Δ-free process**
  In each step: add one random edge that does not close a Δ
- **Semi-random variation of Δ-free process** (Rödl nibble type)
  In each step: add "many" random-like edges that do not close a Δ

G., Warnke (2020): almost packing of nearly optimal $R(3,t)$ graphs

Given $\epsilon > 0$, we find edge-disjoint $(G_i)_{i\in\mathcal{I}}$ with $G_i \subseteq K_n$ such that
(a) each n-vertex graph G_i is Δ-free with $\alpha(G_i) \le C_\epsilon\sqrt{n \log n}$
(b) $|\bigcup_{i\in\mathcal{I}} E(G_i)| \ge (1-\epsilon)\binom{n}{2}$

- **Via simple polynomial-time randomized algorithm**:
  Sequentially choose G_i via semi-random variation of Δ-free process
- **Main technical challenge**:
  Controlling errors over $\Theta(\sqrt{n/\log n})$ iterations of the process

Proof: High-level Strategy/Ideas

Main technical result: find random-like Δ-free subgraph $G \subseteq H$

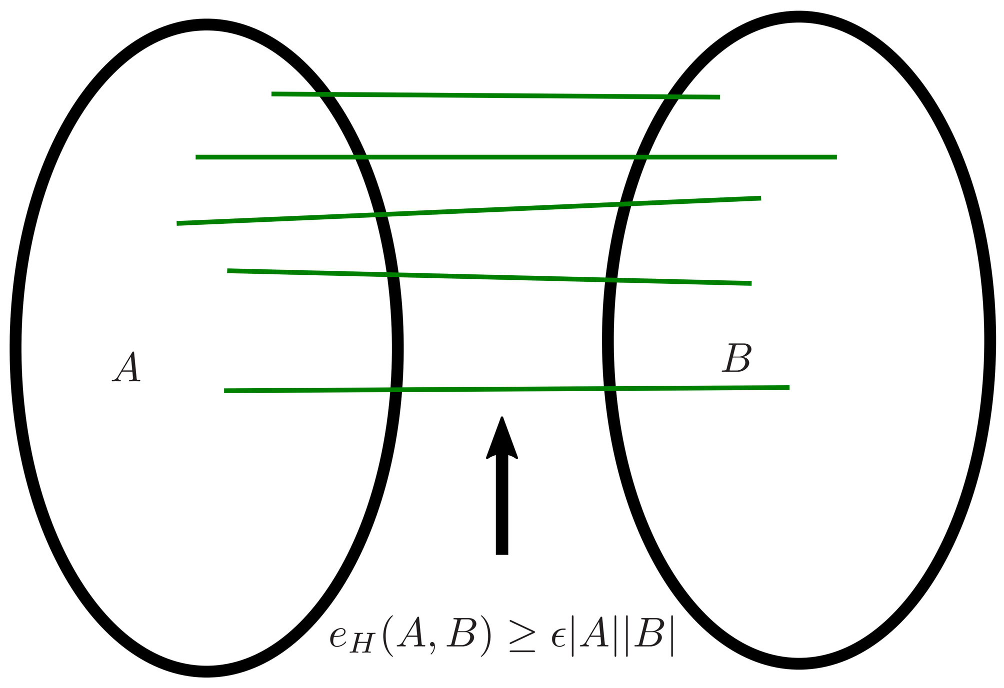
Let $p := \sqrt{\beta(\log n)/n}$ and $s := C_\epsilon\sqrt{n \log n}$. If $H \subseteq K_n$ is such that

$$e_H(A,B) \ge \epsilon\,|A||B|$$

for all disjoint sets A, B of size s, then we can find Δ-free $G \subseteq H$ with

$$e_G(A,B) = (1 \pm \delta)\, p\, e_H(A,B)$$

for all disjoint sets A, B of size s.



Proof based on semi-random variation of Δ-free process:

- *Crucial that we do not require degree/codegree regularity of H*
- "Self-stabilization" mechanism built into process (to control errors)
- One-sided error estimates (only track large-sets two-sided)
- Tools: Bounded-Differences-Ineq., and Upper-Tail-Ineq. of Warnke

Algorithm for packing result: *(maintain $e_{H_i}(A,B)$ bounds inductively)*

- Start with $H_0 = K_n$
- Sequentially choose $G_i \subseteq H_i$ and set $H_{i+1} = H_i \setminus G_i$, noting
  $e_{H_i}(A,B) = (1-(1\pm\delta)p)^i|A||B|$
- Stop when $e_{H_I}(A,B) \approx \epsilon|A||B|$ holds

Application: Solve Ramsey-Conjecture of Fox et al.

Ramsey-Definitions for $r \ge 2$ colors

- $\mathbf{G} \to (\mathbf{K_k})_\mathbf{r}$ if any r-coloring of $E(G)$ has monochromatic copy of K_k
- $\mathcal{M}(\mathbf{k},\mathbf{r})$:= all graphs G that are r-Ramsey minimal for K_k
  (i.e., $G \to (K_k)_r$ and $G' \not\to (K_k)_r$ for all $G' \subsetneq G$)

Classical Ramsey-Parameters

- $\min_{G\in\mathcal{M}(k,2)} v(G)$ = Ramsey number $R(k)$
- $\min_{G\in\mathcal{M}(k,2)} e(G)$ = Size Ramsey number $\hat{R}(k)$

Minimum degree version (Burr-Erdős-Lovász 1976)

- $\min_{G\in\mathcal{M}(k,r)} \delta(G) =: s(k,r)$

Previous Results

- $s(k,2) = (k-1)^2$: Burr-Erdős-Lovász (1976)
- $s(k,r) = \tilde{\Theta}_k(r^2)$: Fox-Grinshpun-Liebenau-Person-Szabó (2015)

Fox-Grinshpun-Liebenau-Person-Szabó (JCTB, 2015)

$$cr^2\log r \le s(3,r) \le Cr^2(\log r)^2$$

- They optimized their $s(k,r) = \tilde{\Theta}_k(r^2)$ bounds for $k=3$
- Upper bound: they pack G_i sequentially via LLL–argument

Conjecture (Fox-Grinshpun-Liebenau-Person-Szabó, 2015)

$$s(3,r) = O(r^2\log r)$$

- Hope: maybe can pack G_i sequentially via Δ-free process?
- For technical reasons (to make error-analysis tractable):
  We pack G_i sequentially via *semi-random* Δ-free process

G., Warnke (2020): our packing result implies

$$s(3,r) = \Theta(r^2\log r)$$

Reference: He Guo and Lutz Warnke, *Packing nearly optimal Ramsey $R(3,t)$ graphs*, ***Combinatorica*** **40**, 63–103 (2020)