COLORING THE INTERSECTION OF TWO MATROIDS

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Abstract. A result [The intersection of a matroid and a simplicial complex, *Trans. Amer. Math. Soc.* **358**] from 2006 of Aharoni and the first author of this paper states that for any two positive integers p, q, where p divides q, if a matroid $\mathcal{M}$ is p-colorable and a matroid $\mathcal{N}$ is q-colorable then $\mathcal{M} \cap \mathcal{N}$ is $(p+q)$-colorable. In this paper we show that the assumption that p divides q is in fact redundant, and we also prove that $\mathcal{M} \cap \mathcal{N}$ is even $p+q$ list-colorable.

The result uses topology and relies on a new parameter yielding a lower bound for the topological connectivity of the intersection of two matroids.

1. Introduction

A *hypergraph* is a pair $H = (V, E)$ where the *vertex set* V is a finite set and the *edge set* E is a set of subsets of V. A set $X \subseteq V$ is called *independent* in H if there is no $e \in E$ such that $e \subseteq X$. We denote by $\mathcal{I}(H)$ the set of all independent sets in H.

A property of a set of the form $\mathcal{C} = \mathcal{I}(H)$ is that if $T \in \mathcal{C}$ and $S \subseteq T$ then $S \in \mathcal{C}$. For a finite set V, a set $\mathcal{C}$ of subsets of V with such closed under taking subsets property is called an *(abstract simplicial) complex*, and V is called the *ground set* of the complex $\mathcal{C}$. For convenience, in this note we assume that every element in the ground set of a complex is included in some set of the complex.

Definition 1.1. *A complex $\mathcal{M}$ is called a matroid if the following hold:*

- $\emptyset \in \mathcal{M}$.
- *(Independence augmentation axiom) If $S, T \in \mathcal{M}$ and $|S| < |T|$, then there exists $v \in T \setminus S$ such that $S \cup \{v\} \in \mathcal{M}$.*

A set in $\mathcal{M}$ is called an independent set of the matroid $\mathcal{M}$.

Two widely studied parameters in graph theory, chromatic number and list chromatic number of a graph G, can be generalized to a complex $\mathcal{C}$, as defined in the following way (in the former case, $\mathcal{C} = \mathcal{I}(G)$).

Definition 1.2. *Given a complex $\mathcal{C}$ on the ground set V, the chromatic number $\chi(\mathcal{C})$ of $\mathcal{C}$ is the minimum number of sets in $\mathcal{C}$ such that their union is V.*

Definition 1.3. *Given a complex $\mathcal{C}$ on the ground set V, the list chromatic number $\chi_\ell(\mathcal{C})$ of $\mathcal{C}$ is the minimum number k such that for any lists of colors $(L_v)_{v \in V}$ of size k, there exists a choice function $f : V \to \cup_{v \in V} L_v$ such that $f(v) \in L_v$ for every v and $f^{-1}(c) \in \mathcal{C}$ for every color c.*

Setting $L_v = \{1, \ldots, \chi_\ell(\mathcal{C})\}$ for each $v \in V$ proves

$$\chi(\mathcal{C}) \le \chi_\ell(\mathcal{C}).$$

In [1], it is proved that for two matroids $\mathcal{M}$ and $\mathcal{N}$ on the same ground set, then

$$\chi(\mathcal{M} \cap \mathcal{N}) \le 2 \max(\chi(\mathcal{M}), \chi(\mathcal{N})).$$

And it is also proved that if $\chi(\mathcal{M})$ divides $\chi(\mathcal{N})$, then

$$\chi(\mathcal{M} \cap \mathcal{N}) \le \chi(\mathcal{M}) + \chi(\mathcal{N}).$$

In this paper, we extend these results.

Theorem 1.4. *For two matroids $\mathcal{M}$ and $\mathcal{N}$ on the same ground set,*

$$\chi(\mathcal{M} \cap \mathcal{N}) \le \chi(\mathcal{M}) + \chi(\mathcal{N}).$$

Together with a recent result in [2], our proof also leads to the same upper bound on the list chromatic number.

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Theorem 1.5. *For two matroids $\mathcal{M}$ and $\mathcal{N}$ on the same ground set,*

$$\chi_\ell(\mathcal{M} \cap \mathcal{N}) \leq \chi(\mathcal{M}) + \chi(\mathcal{N}).$$

The structure of the paper is the following: in Section 2 we introduce some notation to state our results later. In Section 3 we find a lower bound on the topological connectivity (see Theorem 3.5). In Section 4, we introduce a combinatorial parameter. In Section 5 we find a connection between the topological connectivity and the combinatorial parameter (see Theorem 5.1). In Section 6, we prove Theorem 1.5 by using the relations between (list) chromatic number, topological connectivity, and the combinatorial parameter.

2. Circuit representation of a matroid and some hypergraph operations

The reversed operator for $\mathcal{I}$ shown in Section 1 can be defined as follows.

Definition 2.1. *Let $\mathcal{C}$ be a complex on the ground set V. We write*

$$circ(\mathcal{C}) = \{e \subseteq V : e \notin \mathcal{C}, \text{ and for every } x \in e,\ e \setminus \{x\} \in \mathcal{C}\}.$$

Observation 2.2. *If $\mathcal{C}$ is a complex on V then $\mathcal{C} = \mathcal{I}((V, circ(\mathcal{C})))$.*

For a matroid $\mathcal{M}$ on the ground set V, it is well-known that for the hypergraph $H = (V, E)$ with $E = circ(\mathcal{M})$, the following properties hold:

- $\emptyset \notin E$,
- there are no two distinct edges $C_1, C_2 \in E$ such that $C_1 \subseteq C_2$, and
- (circuit elimination property) for every two edges $C_1, C_2 \in E$ and every $u \in C_1 \cap C_2$ and $v \in C_1 \setminus C_2$ there exists $C_3 \in E$, such that $C_3 \subseteq C_1 \cup C_2$ and $v \in C_3$ and $u \notin C_3$.

We call such H a *circuit representation* of the matroid $\mathcal{M}$, and an element of $E = circ(\mathcal{M})$ is called a *circuit* of the matroid $\mathcal{M}$. On the other hand, it is well-known that for any hypergraph $H = (V, E)$ satisfying the above properties, $\mathcal{I}(H)$ is a matroid, of which H is a circuit representation.

Definition 2.3. *Let $H = (V, E)$ be a hypergraph. For an edge $e \in E$, we write*

$$H - e = (V, E \setminus \{e\}).$$

For a set $X \subseteq V$, we write

$$\begin{aligned} H[X] &= (X,\ \{e \in E \mid e \subseteq X\}), \\ H/X &= (V \setminus X, \{e \setminus X \mid e \in E,\ e \not\subseteq X\}), \\ H \setminus X &= (V \setminus X,\ \{e \in E \mid e \cap X = \emptyset\}), \\ H \sim X &= (V,\ \{e \in E \mid e \cap X = \emptyset\}). \end{aligned}$$

If $v \in V$ then we write $H \sim v = H \sim \{v\}$.

Note that $H \setminus X$ and $H \sim X$ differ only by their vertex sets, and $H[X] = H \setminus (V \setminus X)$.

When applying the operators $[\quad]$, $/$, and $\sim$ to a matroid, we refer to the application of these operators on the circuit representation of the matroid. Formally, if $\mathcal{M}$ is a matroid on the ground set V, whose circuit representation is $H = (V, circ(\mathcal{M}))$, and if $X \subseteq V$, then we define

$$\mathcal{M}[X] = \mathcal{I}(H[X]), \quad \mathcal{M}/X = \mathcal{I}(H/X), \quad \text{and} \quad \mathcal{M} \sim X = \mathcal{I}(H \sim X),$$

and if $v \in V$ then we define

$$\mathcal{M} \sim v = \mathcal{I}(H \sim v).$$

Note that $\mathcal{M}[X]$, $\mathcal{M}/X$, and $\mathcal{M} \sim X$ in the above definitions are matroids, and they coincide with the usual definitions of the restriction of $\mathcal{M}$ to X, the contraction of X from $\mathcal{M}$, and the join (direct sum) $2^X * \mathcal{M}[V \setminus X]$ (see Section 3 for the definition), respectively.

3. The topological parameter η

Let $\mathcal{C}$ be an abstract simplicial complex. Assuming we fix some ring R, we can apply homology theory on $\mathcal{C}$. We write *(homological) connectivity* $\eta(\mathcal{C})$ for the minimal value of k such that the reduced homology $\tilde{H}_{k-1}(\mathcal{C}, R)$ does not vanish. If $\mathcal{C}$ is the empty complex then we write $\eta(\mathcal{C}) = 0$ and if all reduced homology groups of $\mathcal{C}$ vanish then we write $\eta(\mathcal{C}) = \infty$. See, e.g., [1, Section 2], for the geometric meaning of the connectivity.

For two complexes $\mathcal{C}, \mathcal{D}$ on disjoint sets, the *join* $\mathcal{C} * \mathcal{D}$ is $\{S \cup T \mid S \in \mathcal{C}, T \in \mathcal{D}\}$.

Theorem 3.1. [1] *Let $\mathcal{C}$ and $\mathcal{D}$ be complexes on disjoint sets. Then*

$$\eta(\mathcal{C} * \mathcal{D}) \geq \eta(\mathcal{C}) + \eta(\mathcal{D}).$$

For two complexes on the same set, we can prove the following inequalities using the Mayer–Vietoris sequence:

Theorem 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two abstract simplicial complexes on the same set V. Then*

(1) $\eta(\mathcal{A} \cup \mathcal{B}) \geq \min\Big(\eta(\mathcal{A}),\ \eta(\mathcal{B}),\ \eta(\mathcal{A} \cap \mathcal{B}) + 1\Big)$.

(2) $\eta(\mathcal{A} \cap \mathcal{B}) \geq \min\Big(\eta(\mathcal{A}),\ \eta(\mathcal{B}),\ \eta(\mathcal{A} \cup \mathcal{B}) - 1\Big)$.

(3) $\eta(\mathcal{A}) \geq \min\Big(\eta(\mathcal{A} \cup \mathcal{B}),\ \eta(\mathcal{A} \cap \mathcal{B})\Big)$.

Using inequality (1) of Theorem 3.2 for $\mathcal{A} = \mathcal{I}(H \setminus \{v\})$ and $\mathcal{B} = 2^{\{v\}} * \mathcal{I}\Big(H \setminus \big(\{v\} \cup N_H(v)\big)\Big)$, where 2^S is the power set of the set S and $N_H(v)$ is the set of neighbors of v in H, we can deduce the following:

Theorem 3.3. *Let $H = (V, E)$ be a hypergraph and let v be a vertex of H such that $\{v\} \notin E$. Then*

$$\eta(\mathcal{I}(H)) \geq \min\Big(\eta\big(\mathcal{I}(H \setminus \{v\})\big),\ \eta\big(\mathcal{I}(H/\{v\})\big) + 1\Big).$$

Using inequality (3) of Theorem 3.2 for $\mathcal{A} = \mathcal{I}(H)$ and $\mathcal{B} = 2^e * \mathcal{I}(H/e)$ we can also deduce the following (see [2, Section 8.3] for the details of the proof):

Theorem 3.4. *Let H be a hypergraph and let e be an edge of H which does not contain any other edge. Then*

$$\eta(\mathcal{I}(H)) \geq \min\Big(\eta\big(\mathcal{I}(H - e)\big),\ \eta\big(\mathcal{I}(H/e)\big) + |e| - 1\Big).$$

Theorem 3.4 was proved in [4] for the case that H is a graph, but the same proof holds for general hypergraphs as well. While Theorem 3.4 is extensively used for graphs, its use for hypergraphs is less common so far. It is used implicitly in [3].

Repeatedly applying Theorem 3.3 and Theorem 3.4 gives a very powerful tool for obtaining lower bounds for η.

Theorem 3.5. *Let $\mathcal{M}_1, \ldots, \mathcal{M}_k$ be matroids on the ground set V, and let $v \in V$. Then either*

$$\eta(\mathcal{M}_1 \cap \ldots \cap \mathcal{M}_k) \geq \eta\Big((\mathcal{M}_1 \sim v) \cap \mathcal{M}_2 \cap \ldots \cap \mathcal{M}_k\Big)$$

or there exists C such that

(i) $v \in C$.

(ii) C *is a circuit of* $\mathcal{M}_1$.

(iii) $C \in \bigcap_{i=2}^k \mathcal{M}_i$.

(iv) $\eta(\mathcal{M}_1 \cap \ldots \cap \mathcal{M}_k) \geq \eta\Big((\mathcal{M}_1/C) \cap \ldots \cap (\mathcal{M}_k/C)\Big) + |C| - 1$.

Proof. Let $H_i = (V, E_i) = (V, circ(\mathcal{M}_i))$ be the circuit representation of $\mathcal{M}_i$ for $i = 1, \ldots, k$. For $I \subseteq \{1, \ldots, k\}$, let $\cup_{i \in I} H_i = (V, \cup_{i \in I} E_i)$ and let $H = \cup_{i=1}^k H_i$. So $\mathcal{I}(H) = \cap_{i=1}^k \mathcal{M}_i$.

Let $C_1, \ldots, C_t$ be all the circuits of $\mathcal{M}_1$ satisfying that $v \in C_j$ and $C_j \in \cap_{i=2}^k \mathcal{M}_i$ for each $1 \leq j \leq t$.

Claim 3.6.

$$\mathcal{I}(H - C_1 - \cdots - C_t) = (\mathcal{M}_1 \sim v) \cap \mathcal{M}_2 \cap \cdots \cap \mathcal{M}_k, \tag{1}$$

$$\mathcal{I}\big((H - C_1 - \cdots - C_{j-1})/C_j\big) = \mathcal{I}(H/C_j) \quad \textit{for each } 1 \leq j \leq t, \tag{2}$$

$$\mathcal{I}(H/C_j) = (\mathcal{M}_1/C_j) \cap \cdots \cap (\mathcal{M}_k/C_j) \quad \textit{for each } 1 \leq j \leq t. \tag{3}$$

Suppose the claim is true, then applying Theorem 3.4 repeatedly, where each C_j in turn takes the role of e in the theorem, yields either by (1) that

$$\eta(\mathcal{I}(H)) \geq \eta\Big(\mathcal{I}(H - C_1 - \cdots - C_t)\Big) = \eta\Big((\mathcal{M}_1 \sim v) \cap \mathcal{M}_2 \cap \cdots \cap \mathcal{M}_k\Big),$$

or for some $1 \le j \le t$ by (2) and (3) that

$$\begin{aligned}\eta(\mathcal{I}(H)) &\ge \eta\Big(\mathcal{I}\big((H - C_1 - \cdots - C_{j-1})/C_j\big)\Big) + |C_j| - 1\\ &= \eta\Big(\mathcal{I}(H/C_j)\Big) + |C_j| - 1\\ &= \eta\Big((\mathcal{M}_1/C_j) \cap \cdots \cap (\mathcal{M}_k/C_j)\Big) + |C_j| - 1,\end{aligned}$$

in which case setting $C = C_j$ completes the proof. □

Proof of the claim. To verify (1), for S in the LHS, since for each $1 \le j \le t$, $C_j \in \cap_{i=2}^k \mathcal{M}_i$, which implies that C_j is not an edge of $\cup_{i=2}^k H_i$, then S does not contain any edge of $\cup_{i=2}^k H_i$, therefore $S \in \cap_{i=2}^k \mathcal{M}_i$. And $S \setminus \{v\}$ does not contain any edge $f \in H_1$ satisfying $f \subseteq V \setminus \{v\}$, since such $f \neq C_j$ for every $1 \le j \le t$. Therefore $S \setminus \{v\} \in \mathcal{M}_1[V \setminus \{v\}]$ and $S \in \mathcal{M}_1 \sim v$. Thus the LHS is contained in the RHS. For T in the RHS, $T \in \cap_{i=2}^k \mathcal{M}_i$ implies that T is independent in $\cup_{i=2}^k H_i$. And T does not contain any edge of $H_1[V \setminus \{v\}]$. Furthermore, we claim that T does not contain any $C \in H_1$ such that $v \in C$ and $C \neq C_j$ for every $1 \le j \le t$: suppose not, then such C must contain an edge of some H_i for $2 \le i \le k$ as $C_1, \ldots, C_t$ are all the edges of H_1 including v and containing no edge of $\cup_{i=2}^t H_i$, which contradicts to the assumption $T \in \mathcal{M}_i$. Therefore $T \in \mathcal{I}(H - C_1 - \cdots - C_t)$ and then the RHS is contained in the LHS.

To verify (2), the RHS is contained in the LHS, since $(H - C_1 - \cdots - C_{j-1})/C_j$ and H/C_j have the same vertex set and the edge set of the former is contained in that of the latter. On the other hand, for any S in the LHS, we claim that S does not contain $C_\ell \setminus C_j$ for any $1 \le \ell \le j-1$: suppose not, i.e., $C_\ell \setminus C_j \subseteq S$ for some $1 \le \ell \le j-1$. Since $v \in C_j \cap C_\ell$, then by the circuit elimination property, there exists a circuit C' of $\mathcal{M}_1$ such that $C' \subseteq (C_j \cup C_\ell) \setminus \{v\}$. Then S contains $C' \setminus C_j$, but C' is an edge of H different from any C_ℓ for $1 \le \ell \le j-1$ (since $v \notin C'$), which contradicts with the assumption that $S \in \mathcal{I}\big((H - C_1 - \cdots - C_{j-1})/C_j\big)$. Therefore the claim holds and $S \in \mathcal{I}(H/C_j)$, which proves that the LHS is contained in the RHS.

To verify (3), since $C_j \in \cap_{i=2}^k \mathcal{M}_i$, no edge of $\cup_{i=2}^k H_i$ is contained in C_j. And since $C_j \in circ(\mathcal{M}_1)$, no edge of H_1 other than C_j itself is contained in C_j. Therefore

$$H/C_j = (\cup_{i=1}^k H_i)/C_j = \cup_{i=1}^k (H_i/C_j). \tag{4}$$

By definition $\mathcal{I}(H_i/C_j) = \mathcal{M}_i/C_j$ for each $i = 1, \ldots, k$, therefore together with (4), $\mathcal{I}(H/C_j) = \cap_{i=1}^k \mathcal{I}(H_i/C_j) = \cap_{i=1}^k (\mathcal{M}_i/C_j)$. □

4. The combinatorial parameter $\nu_{p,q}(\mathcal{M}, \mathcal{N})$

For sets $A_1, \ldots, A_k$ and an element v, we write $\#v(A_1, \ldots, A_k)$ for the number of sets among $A_1, \ldots, A_k$ to which v belongs. For two positive integers p, q and two matroids $\mathcal{M}, \mathcal{N}$ on the same ground set V we define

$$\nu_{p,q}(\mathcal{M}, \mathcal{N}) = \max_{\substack{A_1,\ldots,A_p \in \mathcal{M}\\ B_1,\ldots,B_q \in \mathcal{N}}} \sum_{v \in V} \min\big(\#v(A_1, \ldots, A_p),\ \#v(B_1, \ldots, B_q)\big).$$

Note that $\nu_{1,1}(\mathcal{M}, \mathcal{N})$ is the size of the largest set which is independent in both $\mathcal{M}$ and $\mathcal{N}$.

Observation 4.1. *The parameter ν is monotone in p, q, $\mathcal{M}$ and $\mathcal{N}$, i.e., if p, q, p', q' are positive integers and $\mathcal{M}, \mathcal{N}, \mathcal{M}', \mathcal{N}'$ are matroids such that $p \le p'$, $q \le q'$, $\mathcal{M} \subseteq \mathcal{M}'$ and $\mathcal{N} \subseteq \mathcal{N}'$ then $\nu_{p,q}(\mathcal{M}, \mathcal{N}) \le \nu_{p',q'}(\mathcal{M}', \mathcal{N}')$*

By the closed down property of $\mathcal{M}$ and $\mathcal{N}$ and the double-counting argument, we can get the following observation.

Observation 4.2. *For any two positive integers p, q and two matroids $\mathcal{M}, \mathcal{N}$ on the same ground set V, there exist sets $A_1, \ldots, A_p \in \mathcal{M}$ and $B_1, \ldots, B_q \in \mathcal{N}$ such that $\sum_{i=1}^p |A_i| = \sum_{j=1}^q |B_j| = \nu_{p,q}(\mathcal{M}, \mathcal{N})$ and every $v \in V$ satisfies $\#v(A_1, \ldots, A_p) = \#v(B_1, \ldots, B_q)$.*

Lemma 4.3. *If $\mathcal{M}, \mathcal{N}$ are matroids on a common ground set, then for every positive integer q*

$$\nu_{1,1}(\mathcal{M}, \mathcal{N}) \ge \lceil \frac{\nu_{q,q}(\mathcal{M}, \mathcal{N})}{q} \rceil.$$

Proof. Let V be the common ground set. By Edmonds' matroid intersection theorem there exist sets $I, V_1, V_2 \subseteq V$ such that $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$ and $I \in \mathcal{M} \cap \mathcal{N}$ and $I \cap V_1$ is the largest among subsets of V_1 which are independent in $\mathcal{M}$, and $I \cap V_2$ is the largest among subsets of V_2 which are independent in $\mathcal{N}$.

Now for every $A_1, \dots, A_q \in \mathcal{M}$ and $B_1, \dots, B_q \in \mathcal{N}$ we have

$$\begin{aligned}&\sum_{v\in V} \min\left(\#v(A_1,\dots,A_q),\ \#v(B_1,\dots,B_q)\right)\\ =&\sum_{v\in V_1} \min\left(\#v(A_1,\dots,A_q),\ \#v(B_1,\dots,B_q)\right) + \sum_{v\in V_2} \min\left(\#v(A_1,\dots,A_q),\ \#v(B_1,\dots,B_q)\right)\\ \le&\sum_{v\in V_1} \#v(A_1\cap V_1,\dots,A_q\cap V_1) + \sum_{v\in V_2} \#v(B_1\cap V_2,\dots,B_q\cap V_2)\\ \le& q\cdot|I\cap V_1| + q\cdot|I\cap V_2| = q|I| \le q\nu_{1,1}(\mathcal{M},\mathcal{N}),\end{aligned}$$

which completes the proof. ☐

Lemma 4.4. *Let $\mathcal{M}, \mathcal{N}$ be two matroids on the same ground set V such that $\nu_{1,1}(\mathcal{M},\mathcal{N}) > 0$. Let p, q be positive integers with $p \le q$. Then there exist sets $X_1, \dots X_p \in \mathcal{M}$ and $Y_1, \dots, Y_q \in \mathcal{N}$ and an element $z \in V$ with the following properties:*

- *Every $v \in V \setminus \{z\}$ satisfies $\#v(X_1,\dots,X_p) = \#v(Y_1,\dots,Y_q)$,*
- *$\#z(X_1,\dots,X_p) = p$,*
- *$\sum_{i=1}^q |Y_i| = \nu_{p,q}(\mathcal{M},\mathcal{N})$.*

Proof. Write $r = \nu_{1,1}(\mathcal{M},\mathcal{N})$. Let $Z \in \mathcal{M}\cap\mathcal{N}$ have size r, and let $A_1, \dots, A_p \in \mathcal{M}$ and $B_1, \dots, B_q \in \mathcal{N}$ be chosen such that

(a) $\sum_{v\in V} \min\left(\#v(A_1,\dots,A_p),\ \#v(B_1,\dots,B_q)\right) = \nu_{p,q}(\mathcal{M},\mathcal{N}) - 1$,
(b) Subject to the above condition $\sum_{v\in Z} \min\left(\#v(A_1,\dots,A_p),\ \#v(B_1,\dots,B_q)\right)$ is maximal,
(c) Subject to the above two conditions $\sum_{i=1}^p |A_i| + \sum_{j=1}^q |B_j|$ is minimal.

Note that $\sum_{i=1}^p |A_i| + \sum_{j=1}^q |B_j|$ is minimal in condition (c) guarantees that for every $v \in V$,

$$\#v(A_1,\dots,A_p) = \#v(B_1,\dots,B_q), \tag{5}$$

since if $\#v(A_1,\dots,A_p) > \#v(B_1,\dots,B_q)$ for some $v \in V$, we can remove v from some of A_i including it, which does not violate condition (a) or (b), but has a smaller total size, a contradiction. Similarly, we can get a contradiction if $\#v(A_1,\dots,A_p) < \#v(B_1,\dots,B_q)$ for some $v \in V$.

Therefore we have

$$\sum_{j=1}^q |B_j| = \sum_{v\in V} \#v(B_1,\dots,B_q) = \nu_{p,q}(\mathcal{M},\mathcal{N}) - 1 < \nu_{p,q}(\mathcal{M},\mathcal{N}) \underset{\textit{Observation } 4.1}{\le} \nu_{q,q}(\mathcal{M},\mathcal{N}) \underset{\textit{Lemma } 4.3}{\le} qr,$$

which implies at least one set among $B_1, \dots, B_q$ has size less than r. Without loss of generality $|B_q| < r$. By the independence augmentation axiom, this implies that for some $z \in Z \setminus B_q$ we have $B'_q = B_q \cup \{z\} \in \mathcal{N}$.

We now claim that for every $i \in \{1, \dots, p\}$ we have $A_i \cup \{z\} \in \mathcal{M}$. Suppose not, then, say, $A_p \cup \{z\} \notin \mathcal{M}$, so it contains some circuit C of $\mathcal{M}$ and $z \in C$. Take some $c \in C \setminus Z$. Then $c \in A_p$. By the independence augmentation axiom, $C \setminus \{c\} \in \mathcal{M}$ can be extended to a size $|A_p|$ set $A'_p = (A_p \cup \{z\}) \setminus \{c\} \in \mathcal{M}$. Since

$$\#c(A_1,\dots,A_{p-1},A'_p) = \#c(A_1,\dots,A_{p-1},A_p) - 1$$

and

$$\begin{aligned}&\min\Big(\#z(A_1,\dots,A_{p-1},A'_p),\ \#z(B_1,\dots,B_{q-1},B'_q)\Big)\\ &= \min\Big(\#z(A_1,\dots,A_p),\ \#z(B_1,\dots,B_q)\Big) + 1,\end{aligned}$$

then by (5) we now have

$$\begin{aligned}&\sum_{v\in V} \min\Big(\#v(A_1,\dots,A_{p-1},A'_p),\ \#v(B_1,\dots,B_{q-1},B'_q)\Big)\\ =&\sum_{v\in V} \min\Big(\#v(A_1,\dots,A_{p-1},A_p),\ \#v(B_1,\dots,B_{q-1},B_q)\Big) = \nu_{p,q}(\mathcal{M},\mathcal{N}) - 1\end{aligned}$$

and

$$\sum_{v\in Z} \min\Big(\#v(A_1,\dots,A_{p-1},A'_p),\ \#v(B_1,\dots,B_{q-1},B'_q)\Big)$$
$$= \sum_{v\in Z} \min\Big(\#v(A_1,\dots,A_p),\ \#v(B_1,\dots,B_q)\Big) + 1$$

contradicting the way (b) in which $A_1,\dots,A_q$ and $B_1,\dots,B_q$ were chosen. This finishes the proof of the claim.

Now the sets $X_i = A_i \cup \{z\}$ (for $i = 1,\dots,p$) and $Y_j = B_j$ (for $j = 1,\dots,q-1$) and $Y_q = B_q \cup \{z\}$ satisfy the required conditions of the lemma. □

5. Relation between the parameters

Theorem 5.1. *Let p, q be two positive integers and let $\mathcal{M}, \mathcal{N}$ be two matroids on the same ground set. Then*

$$\eta(\mathcal{M} \cap \mathcal{N}) \geq \frac{\nu_{p,q}(\mathcal{M},\mathcal{N})}{p+q}\ .$$

Proof. The proof is by induction on the size of the common ground set V. When $V = \emptyset$, $\eta(\mathcal{M}\cap\mathcal{N}) = \nu_{p,q}(\mathcal{M},\mathcal{N}) = 0$. The statement is true.

Next we turn to the case $|V| \geq 1$. We assume without loss of generality $p \leq q$ and by Lemma 4.4, there exist sets $X_1,\dots X_p \in \mathcal{M}$ and $Y_1,\dots,Y_q \in \mathcal{N}$ and an element $z \in V$ with the following properties:

- Every $v \in V \setminus \{z\}$ satisfies $\#v(X_1,\dots,X_p) = \#v(Y_1,\dots,Y_q)$,
- $\#z(X_1,\dots,X_p) = p$,
- $\sum_{i=1}^{q} |Y_i| = \nu_{p,q}(\mathcal{M},\mathcal{N})$.

We apply Theorem 3.5 with $\mathcal{M}$ taking the role of $\mathcal{M}_1$. Then either

$$\eta(\mathcal{M}\cap\mathcal{N}) \geq \eta\Big((\mathcal{M}\sim z)\cap\mathcal{N}\Big) \tag{6}$$

or there is some circuit C in $\mathcal{M}$ such that $z \in C$, $C \in \mathcal{N}$, and

$$\eta(\mathcal{M}\cap\mathcal{N}) \geq \eta\Big((\mathcal{M}/C)\cap(\mathcal{N}/C)\Big) + |C| - 1. \tag{7}$$

If (7) occurs, write $I = C \setminus \{z\}$ and $s = |I| = |C| - 1$. Let $A_1,\dots,A_p \in \mathcal{M}$ and $B_1,\dots,B_q \in \mathcal{N}$ satisfy $\nu_{p,q}(\mathcal{M},\mathcal{N}) = \sum_{u\in V} \min\Big(\#u(A_1,\dots,A_p),\ \#u(B_1,\dots,B_q)\Big)$.

We can find sets $S_1,\dots S_p, T_1,\dots,T_q$, each of size s, such that $A_i \setminus S_i \in \mathcal{M}/C$ for all $i \in \{1,\dots,p\}$ and $(B_j \setminus \{z\}) \setminus T_j \in \mathcal{N}/C$ for all $j \in \{1,\dots,q\}$. In detail, to construct S_i, since C is a circuit in $\mathcal{M}$, $C \not\subseteq A_i$ and then $|A_i \cap C| < |C| = s+1$, we take $A_i \cap C$ and add any $s - |A_i \cap C|$ other elements. To construct T_i, we take $(B_j \setminus \{z\}) \cap C$ and add any $s - |(B_j \setminus \{z\}) \cap C|$ other elements.

Write $A'_i = A_i \setminus S_i$, which is in $\mathcal{M}/C$, for all $i \in \{1,\dots,p\}$, and $B'_j = (B_j \setminus \{z\}) \setminus T_j$, which is in $\mathcal{N}/C$, for all $j \in \{1,\dots,q\}$. We now claim that for each $u \in V$ we have

$$\min\Big(\#u(A_1,\dots,A_p),\ \#u(B_1,\dots,B_q)\Big)$$
$$\leq \min\Big(\#u(A'_1,\dots,A'_p),\ \#u(B'_1,\dots,B'_q)\Big) + \#u(S_1,\dots,S_p,T_1,\dots,T_q).$$

Indeed, when $u \neq z$ this is trivial, since $u \in A_i$ if and only if $u \in A'_i$ or S_i, and $u \in B_j$ if and only if $u \in B'_j$ or T_j; and for $u = z$ it follows from that fact that whenever $z \in A_i$ we must have also $z \in S_i$ so that

$$\#z(A_1,\dots,A_p) \leq \#z(S_1,\dots,S_p,T_1,\dots,T_q).$$

We thus have

$$\nu_{p,q}(\mathcal{M},\mathcal{N}) = \sum_{u\in V} \min\Big(\#u(A_1,\dots,A_p),\ \#u(B_1,\dots,B_q)\Big)$$
$$\leq \sum_{u\in V} \left(\min\Big(\#u(A'_1,\dots,A'_p),\ \#u(B'_1,\dots,B'_q)\Big) + \#u(S_1,\dots,S_p,T_1,\dots,T_q)\right)$$
$$\leq \nu_{p,q}(\mathcal{M}/C,\mathcal{N}/C) + (p+q)s$$

and by the induction hypothesis

$$\eta(\mathcal{M}\cap\mathcal{N}) \geq \eta\Big((\mathcal{M}/C)\cap(\mathcal{N}/C)\Big) + s \geq \frac{\nu_{p,q}(\mathcal{M}/C,\mathcal{N}/C)}{p+q} + s \geq \frac{\nu_{p,q}(\mathcal{M},\mathcal{N})}{p+q}.$$

If (6) occurs, applying Theorem 3.5 again to $\mathcal{M}\sim z$ and $\mathcal{N}$ with $\mathcal{N}$ taking the role of $\mathcal{M}_1$, either

$$\eta(\mathcal{M}\cap\mathcal{N}) \geq \eta\Big((\mathcal{M}\sim z)\cap\mathcal{N}\Big) \geq \eta\Big((\mathcal{M}\sim z)\cap(\mathcal{N}\sim z)\Big) \underset{\text{Theorem 3.1}}{\geq} \eta(2^{\{z\}}) = \infty,$$

in which case we have $\eta(\mathcal{M}\cap\mathcal{N}) \geq \frac{\nu_{p+q}(\mathcal{M},\mathcal{N})}{p+q}$, or there exists a circuit D in $\mathcal{N}$ such that $z\in D$ and $D\in\mathcal{M}\sim z$ and

$$\eta(\mathcal{M}\cap\mathcal{N}) \geq \eta\Big((\mathcal{M}\sim z)\cap\mathcal{N}\Big) \geq \eta\Big(((\mathcal{M}\sim z)/D)\cap(\mathcal{N}/D)\Big) + |D|-1.$$

In the last case, similar as above, we can find sets $S'_1,\dots,S'_p,T'_1,\dots,T'_q$, each of size $t=|D|-1$, such that for each $1\leq i\leq p$, $X'_i := (X_i\setminus\{z\})\setminus S'_i \in (\mathcal{M}\sim z)/D$ using the fact that $z\in X_i$, and for each $1\leq j\leq q$, $Y'_j := Y_j\setminus T'_j \in \mathcal{N}/D$. Thus

$$\#u(Y_1,\dots,Y_q) \leq \min\Big(\#u(X'_1,\dots,X'_p),\ \#u(Y'_1,\dots,Y'_q)\Big) + \#u(S'_1,\dots,S'_p,T'_1,\dots,T'_q),$$

since for $u\neq z$, $\#u(Y_1,\dots,Y_q) = \#u(X_1,\dots,X_p)$, and for $u=z$, $z\in Y_j$ implies $z\in T'_j$. Therefore

$$\begin{aligned}\nu_{p,q}(\mathcal{M},\mathcal{N}) &= \sum_{i=1}^{q}|Y_i| = \sum_{u\in V}\#u(Y_1,\dots,Y_q)\\ &\leq \sum_{u\in V}\left(\min\Big(\#u(X'_1,\dots,X'_p),\ \#u(Y'_1,\dots,Y'_q)\Big) + \#u(S'_1,\dots,S'_p,T'_1,\dots,T'_q)\right)\\ &\leq \nu_{p,q}\Big((\mathcal{M}\sim z)/D,\mathcal{N}/D\Big) + (p+q)t,\end{aligned}$$

and by the induction hypothesis,

$$\eta(\mathcal{M}\cap\mathcal{N}) \geq \eta\Big(((\mathcal{M}\sim z)/D)\cap(\mathcal{N}/D)\Big) + t \geq \frac{\nu_{p,q}\Big((\mathcal{M}\sim z)/D,\mathcal{N}/D\Big)}{p+q} + t \geq \frac{\nu_{p,q}(\mathcal{M},\mathcal{N})}{p+q},$$

which completes the proof. □

6. Proof of Theorem 1.5

Notation 6.1. *For a complex $\mathcal{C}$ on the ground set V, let $\Delta_\eta(\mathcal{C}) = \max_{\emptyset\neq S\subseteq V(\mathcal{C})}\frac{|S|}{\eta(\mathcal{C}[S])}$.*

Applying Theorem 4.2 in [1], for a complex $\mathcal{C}$, it is proved in Corollary 8.6 of [1] that $\chi(\mathcal{C}) \leq \lceil\Delta_\eta(\mathcal{C})\rceil$. In [2], this bound is extended to the list chromatic number.

Theorem 6.2. *For a complex $\mathcal{C}$, $\chi_\ell(\mathcal{C}) \leq \lceil\Delta_\eta(\mathcal{C})\rceil$.*

Proof of Theorem 1.5. Let V be the common ground set of $\mathcal{M}$ and $\mathcal{N}$ and let $\mathcal{C}=\mathcal{M}\cap\mathcal{N}$.

Let $p=\chi(\mathcal{M})$ and $q=\chi(\mathcal{N})$. Let $A_1,\dots,A_p\in\mathcal{M}$ satisfying that $\cup_{i=1}^{p}A_i = V$ and let $B_1,\dots,B_q\in\mathcal{N}$ satisfying that $\cup_{j=1}^{q}B_j = V$. Then

$$\min\Big(\#v(A_1,\dots,A_p),\#v(B_1,\dots,B_q)\Big) \geq 1$$

for every $v\in V$, which implies that $\nu_{p,q}(\mathcal{M},\mathcal{N}) \geq |V|$. Thus Theorem 5.1 implies that $\eta(\mathcal{C}) \geq \frac{|V|}{p+q}$, which is equivalent to

$$\frac{|V|}{\eta(\mathcal{C})} \leq p+q.$$

Noting that $\chi(\mathcal{M}[S]) \leq \chi(\mathcal{M})$ and $\chi(\mathcal{N}[S]) \leq \chi(\mathcal{N})$ for every $S\subseteq V$, the above argument works for any non-empty subset S of V, therefore we have

$$\Delta_\eta(\mathcal{C}) \leq \chi(\mathcal{M}) + \chi(\mathcal{N}), \tag{8}$$

which together with Theorem 6.2 completes the proof. □

Remark 6.3. *The bound $\Delta_\eta(\mathcal{M}\cap\mathcal{N}) \leq \chi(\mathcal{M})+\chi(\mathcal{N})$ in (8) is tight.*

Proof. Consider the 4-cycle on $\{1,2,3,4\}$ whose edges[1] are $12, 23, 34, 41$. Then we blow up each of $12, 34$ by p (parallel) edges and blow up each of $23, 41$ by q (parallel) edges. Let the resulting graph be G. Next we define two partition matroids $\mathcal{M}$ and $\mathcal{N}$ on the ground set $E(G)$. The parts of $\mathcal{M}$ are $\Gamma_G(1)$ (all the edges of G incident with vertex 1) and $\Gamma_G(3)$, which satisfies $\chi(\mathcal{M}) = p + q$. The parts of $\mathcal{N}$ are $\Gamma_G(2)$ and $\Gamma_G(4)$, which satisfies $\chi(\mathcal{N}) = p + q$. Let $\mathcal{C}$ be the matching complex of G, i.e., the collection of all the matchings in G. Then $\mathcal{C}$ is the intersection of the two partition matroids $\mathcal{M}$ and $\mathcal{N}$. Since the matching complex has two connected components, $\eta(\mathcal{C}) = 1$ so that

$$\Delta_\eta(\mathcal{M} \cap \mathcal{N}) = |E(G)| = 2(p + q) = \chi(\mathcal{M}) + \chi(\mathcal{N}),$$

where the equality holds. □

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References

[1] R. Aharoni and E. Berger. The intersection of a matroid and a simplicial complex. *Trans. Amer. Math. Soc.* **358** (2006), 4895–4917.

[2] R. Aharoni, E. Berger, H. Guo, and D. Kotlar. Coloring, list coloring, and fractional coloring in intersections of matroids. Preprint (2024). arXiv:2407.08789.

[3] R. Aharoni, E. Berger, and O. Kfir. Acyclic systems of representatives and acyclic colorings of digraphs. *J. Graph Theory* **59** (2008), 177–189.

[4] R. Meshulam. Domination numbers and homology. *J. Combin. Theory Ser. A* **102** (2003), 321–330.

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[1]We use uv as a shorthand for $\{u, v\}$.