Prague dimension of random graphs

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Abstract

The Prague dimension of graphs was introduced by Nešetřil, Pultr and Rödl in the 1970s. Proving a conjecture of Füredi and Kantor, we show that the Prague dimension of the binomial random graph is typically of order $n/\log n$ for constant edge-probabilities. The main new proof ingredient is a Pippenger–Spencer type edge-coloring result for random hypergraphs with large uniformities, i.e., edges of size $O(\log n)$.

1 Introduction

Various notions of dimension are important in many areas of mathematics, as a measure for the complexity of objects. For graphs, one interesting notion of dimension was introduced by Nešetřil, Pultr and Rödl [34, 33] in the 1970s. The Prague dimension $\dim_{\mathrm{P}}(G)$ of a graph G (also called product dimension) is the minimum number d such that G is an induced subgraph of the product of d complete graphs. There are many equivalent definitions of $\dim_{\mathrm{P}}(G)$, see [47, 20, 2], indicating that this is a natural combinatorial notion of dimension [30, 20, 41], which in fact has appealing connections with efficient representations of graphs [47, 25, 18].

Despite receiving considerable attention during the last 40 years (including combinatorial [10, 17], information theoretic [27, 26] and algebraic [34, 30, 1, 2] approaches), the Prague dimension is still not well understood, i.e., its determination usually remains a notoriously[1] difficult task [10, 18]. To gain further insight into the behavior of this intriguing graph parameter, it thus is natural and instructive to investigate the Prague dimension of random graphs, as initiated by Nešetřil and Rödl [35] already in the 1980s. For the binomial random graph $G_{n,p}$, Füredi and Kantor conjectured that with high probability[2] (whp) the order is $\dim_{\mathrm{P}}(G_{n,p}) = \Theta(n/\log n)$ for constant edge-probabilities p, see [18, Conjecture 15] and [24].

In this paper we prove the aforementioned Füredi–Kantor Prague dimension conjecture, by showing that the binomial random graph whp satisfies $\dim_{\mathrm{P}}(G_{n,p}) = \Theta(n/\log n)$ for constant edge-probabilities p.

Theorem 1 (Prague dimension of random graphs)**.** *For any fixed edge-probability $p \in (0,1)$ there are constants $c, C > 0$ so that the Prague dimension of the random graph $G_{n,p}$ satisfies with high probability*

$$c\frac{n}{\log n} \le \dim_{\mathrm{P}}(G_{n,p}) \le C\frac{n}{\log n}. \qquad (1)$$

The Prague dimension of n-vertex graphs can be as large as $n-1$, see [30, 47], so an important consequence of Theorem 1 is that almost all n-vertex graphs have a significantly smaller Prague dimension of order $n/\log n$ (this follows since the random graph $G_{n,1/2}$ is uniformly distributed over all n-vertex graphs).

For our purposes it will be useful to view the Prague dimension as a clique covering and coloring problem. This convenient perspective hinges on the following equivalent definition [47, 2]: that $\dim_{\mathrm{P}}(G)$ equals the minimum number of subgraphs of the complement $\overline{G}$ of G such that (i) each subgraph is a vertex-disjoint union of cliques, and (ii) each edge of $\overline{G}$ is contained in at least one of the subgraphs, but not in all of them.

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[1] The decision problem of whether $\dim_{\mathrm{P}}(G) \le k$ holds is also known to be NP-complete for $k \ge 3$, see [33].

[2] As usual, we say that an event holds *whp* (with high probability) if it holds with probability tending to 1 as $n \to \infty$.

Our main contribution is the upper bound on the Prague dimension in (1), whose proof carefully combines two different random greedy approaches: firstly, a semi-random 'nibble-type' algorithm to iteratively decompose the edges of $\overline{G_{n,p}}$ into edge-disjoint cliques of size $O(\log n)$, and, secondly, a random greedy coloring algorithm to regroup these cliques into $O(n/\log n)$ subgraphs of $\overline{G_{n,p}}$ consisting of vertex-disjoint cliques, which together eventually gives $\dim_{\mathrm{P}}(G_{n,p}) = O(n/\log n)$; see Section 1.3 for more details. Interestingly, this combination allows us to exploit the best features of both greedy approaches: the semi-random approach makes it easier to guarantee certain pseudo-random properties in the first decomposition step, and the random greedy approach makes it easier to guarantee that all cliques are efficiently colored in the second regrouping step (which in fact requires the pseudo-random properties established in the first step).

One major obstacle for this natural proof approach is that the cliques have size $O(\log n)$, which makes many standard tools and techniques unavailable, as they are usually restricted to objects of constant size. Notably, in order to overcome this technical difficulty in the second regrouping step, in this paper we develop a new Pippenger–Spencer type coloring result for random hypergraphs with edges of size $O(\log n)$, which we believe to be of independent interest; see Section 1.1. Beyond Prague dimension and hypergraph coloring results, further contributions of this paper include the proof of a related conjecture of Füredi and Kantor [18], and a strengthening of an old edge-covering result of Frieze and Reed [16]; see Section 1.2.

1.1 Chromatic index of random subhypergraphs

Coloring problems play an important role in much of combinatorics, and in our Prague dimension proof one key ingredient also corresponds to a hypergraph coloring result. The chromatic index $\chi'(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the smallest number of colors needed to properly color its edges, i.e., so that no two intersecting edges receive the same color. Writing $\Delta(\mathcal{H})$ for the maximum degree, it is of fundamental interest to understand when the trivial lower bound $\chi'(\mathcal{H}) \geq \Delta(\mathcal{H})$ is close to the truth. Vizing's theorem from the 1960s states that $\chi'(G) \leq \Delta(G) + 1$ for any graph G. Influential work of Pippenger and Spencer [37] from the 1980s gives a partial answer for r-uniform hypergraphs $\mathcal{H}$ with edges of constant size $r = \Theta(1)$: for any $\delta > 0$ they showed that $\chi'(\mathcal{H}) \leq (1+\delta)\Delta(\mathcal{H})$ for any nearly regular $\mathcal{H}$ with small codegrees, effectively removing the edge-size dependence from the trivial greedy upper bound $\chi'(\mathcal{H}) \leq r(\Delta(\mathcal{H}) - 1) + 1$.

It is challenging to extend the Pippenger–Spencer coloring arguments to edges of size $r = O(\log n)$, which is what we desire in our main Prague dimension proof (where cliques correspond to edges of an auxiliary hypergraph). Our Theorem 2 overcomes this size obstacle in the random setting, i.e., for coloring random edges of any nearly regular hypergraph $\mathcal{H}$ with small codegrees. As we shall see in Sections 1.3 and 2.2.1, this probabilistic Pippenger–Spencer type coloring result indeed suffices for our purposes. Here $\deg_{\mathcal{H}}(v) := |\{e \in E(\mathcal{H}) : v \in e\}|$ and $\deg_{\mathcal{H}}(u,v) := |\{e \in E(\mathcal{H}) : \{u,v\} \subseteq e\}|$ denote the degree and codegree, as usual. Note that in Theorem 2 the maximum degree of $\mathcal{H}_m$ is asymptotic to rm/n, see Remark 3. Since we allow for repeated edges, Theorem 2 also includes an ad-hoc upper bound on the number m of random edges.

Theorem 2 (Chromatic index of random subhypergraphs)**.** *For all reals $\delta, \sigma, b > 0$ with $b \leq \delta\sigma/30$ there is $n_0 = n_0(\delta, \sigma, b) > 0$ such that, for all integers $n \geq n_0$, $2 \leq r \leq b \log n$, $n^{1+\sigma} \leq m \leq n^{rn^{\sigma/5}}$ and all reals $D > 0$, the following holds for every n-vertex r-uniform hypergraph $\mathcal{H}$ satisfying*

$$\max_{v \in V(\mathcal{H})} |\deg_{\mathcal{H}}(v) - D| \leq n^{-\sigma} D \qquad \textit{and} \qquad \max_{u \neq v \in V(\mathcal{H})} \deg_{\mathcal{H}}(u,v) \leq n^{-\sigma} D. \tag{2}$$

We have $\mathbb{P}(\chi'(\mathcal{H}_m) \leq (1+\delta)rm/n) \geq 1 - m^{-\omega(r)}$, where $\mathcal{H}_m$ denotes the random subhypergraph of $\mathcal{H}$ with edges $e_1, \ldots, e_m$, where each edge e_i is independently chosen uniformly at random from $\mathcal{H}$.

Remark 3. *Noting that the expected degree in $\mathcal{H}_m$ is approximately $D \cdot m/|E(\mathcal{H})| = (1+o(1))rm/n \gg r \log m$, for any real $\epsilon > 0$ it is straightforward to see that the maximum degree of $\mathcal{H}_m$ satisfies $\Delta(\mathcal{H}_m) = (1 \pm \epsilon)rm/n$ with probability at least $1 - m^{-\omega(r)}$, say.*

As discussed, for this paper the key point is that Theorem 2 permits edges of size $r = O(\log n)$; we have made no attempt to optimize the ad-hoc assumptions on the number of edges m or the $n^{-\sigma}$ approximation terms in (2). The explicit technical assumption $b \leq \delta\sigma/30$ allows for some flexibility in applications. In particular, for $r = o(\log n)$ we can set $b := \delta\sigma/30$ for any small $\delta > 0$, and for $r = O(\log n)$ we can set $\delta := 30b/\sigma$ for suitable $b > 0$. From these considerations, using Remark 3 we infer that whp

$$\chi'(\mathcal{H}_m) \le \begin{cases} (1+2\delta) \cdot \Delta(\mathcal{H}_m) & \text{if } r = o(\log n), \\ O(1) \cdot \Delta(\mathcal{H}_m) & \text{if } r = O(\log n), \end{cases} \tag{3}$$

which gives Pippenger–Spencer like chromatic index bounds for many non-constant edge-sizes r; we believe that these bounds are of independent interest (see also Corollary 10).

We prove Theorem 2 by showing that a simple random greedy algorithm (that differs from the one used by Pippenger and Spencer [37]) whp produces the desired coloring of the random edges $e_1, \ldots, e_m$ from $\mathcal{H}$. The algorithm we use sequentially assigns each edge e_i a random color in $\{1, \ldots, \lfloor (1+\delta)rm/n \rfloor\}$ that does not appear on some adjacent edge e_j with $j < i$; see Section 3. This random greedy edge coloring algorithm is very natural: Kurauskas and Rybarczyk [29] analyzed it when $\mathcal{H}$ is the complete n-vertex r-uniform hypergraph, and its idea also underpins earlier work that extends the Pippenger–Spencer result to list-colorings [22, 32]. Taking advantage of the random setting, our proof of Theorem 2 uses differential equation method [48, 3, 45] based martingale arguments to show that this greedy algorithm whp properly colors the first m out of $(1+\delta)m$ random edges. This 'more random edges' twist enables us to sidestep some of the 'last few edges' complications[3] that usually arise in the deterministic setting [37, 22, 32], which is one of the reasons why our analysis can allow for edges of size $O(\log n)$; see Section 3 for the details.

1.2 Partitioning the edges of a random graph into cliques

Further motivation for studying the Prague dimension comes from its close connection to the covering and decomposition problems that pervade combinatorics, one interesting non-standard feature being that Theorem 1 requires usage of cliques with $O(\log n)$ vertices, rather than just subgraphs of constant size. The clique covering number $\mathrm{cc}(G)$ of a graph G (also called intersection number) is the minimum number of cliques in G that cover the edge-set of G. Similarly, the clique partition number $\mathrm{cp}(G)$ is the minimum number of cliques in G that partition the edge-set of G. The question of estimating these natural graph parameters was raised by Erdős, Goodman and Pósa [13] in 1966. Motivated in part by applications such as keyword conflicts, traffic phasing and competition graph analysis [36, 28, 39, 9], both $\mathrm{cc}(G)$ and $\mathrm{cp}(G)$ have since been extensively studied for many interesting graph classes, see e.g. [42, 1, 6, 14, 7, 8] and the many references therein.

For random graphs, the study of the clique covering number was initiated in the 1980s by Poljak, Rödl and Turzík [38] and Bollobás, Erdős, Spencer and West [5]. In 1995, Frieze and Reed [16] showed that whp $\mathrm{cc}(G_{n,p}) = \Theta(n^2/(\log n)^2)$ for constant edge-probabilities p. Constructing a clique covering is certainly easier than constructing a clique partition, since it does not have to satisfy such a rigid edge constraint. Indeed, while obviously $\mathrm{cc}(G) \le \mathrm{cp}(G)$, the ratio $\mathrm{cp}(G)/\mathrm{cc}(G)$ can in fact be arbitrarily large, see [12]. However, our Theorem 4 demonstrates that for most graphs the clique partition number and clique covering number have the same order of magnitude.

Theorem 4 (Clique covering and partition number of random graphs)**.** *For every fixed real $\gamma \in (0,1)$ there are constants $c > 0$ and $C = C(\gamma) > 0$ so that if the edge-probability $p = p(n)$ satisfies $n^{-2} \ll p \le 1-\gamma$, then with high probability*

$$c\frac{n^2 p}{(\log_{1/p} n)^2} \le cc(G_{n,p}) \le cp(G_{n,p}) \le C\frac{n^2 p}{(\log_{1/p} n)^2}. \tag{4}$$

The main contribution of (4) is the upper bound, which strengthens the main result of Frieze and Reed [16] from clique coverings to clique partitions, and also allows for $p = p(n) \to 0$. Here the mild assumption $p \le 1-\gamma$ turns out to be necessary, since Lemma 20 implies that whp $\mathrm{cc}(G_{n,p})/(n^2p/(\log_{1/p} n)^2) \to \infty$ as $p \to 1$. The lower bound in (4) is straightforward: it is well-known that $G_{n,p}$ whp has $m = \Theta(n^2 p)$ edges and largest clique of size $\omega = O(\log_{1/p} n)$, which gives $\mathrm{cc}(G_{n,p}) \ge m/\binom{\omega}{2} = \Omega(n^2p/(\log_{1/p} n)^2)$.

To gain a better combinatorial understanding of clique coverings, it is instructive to study and optimize other properties besides the size, such as their thickness $\mathrm{cc}_\Delta(G) := \min_{\mathcal{C}} \Delta(\mathcal{C})$ and chromatic index

[3] Many deterministic approaches such as [37, 22] first efficiently color most of the edges of $\mathcal{H}$ using $(1+\delta/2)\Delta(\mathcal{H})$ colors, say, so that the remaining uncolored 'last few edges' yield a hypergraph with maximum degree at most $\epsilon\Delta(\mathcal{H})$, say. By choosing the constant $\epsilon = \epsilon(r,\delta) > 0$ sufficiently small, these 'last few edges' can then trivially be colored using $r \cdot \epsilon\Delta(\mathcal{H}) \le \delta/2 \cdot \Delta(\mathcal{H})$ additional colors, which clearly becomes harder to implement when $r = r(n) \to \infty$ (as now the dependence of ϵ on r matters).

$\mathrm{cc}'(G) := \min_{\mathcal{C}} \chi'(\mathcal{C})$, where the minimum is taken over all clique coverings $\mathcal{C}$ of the edges of G, i.e., collections $\mathcal{C}$ of cliques in G covering all edges of G (formally thinking of $\mathcal{C}$ as a hypergraph with vertex-set $V(G)$ and edge-set $\mathcal{C}$). Notably, the parameters $\mathrm{cc}'(\overline{G})$ and $\mathrm{cc}_\Delta(\overline{G})$ approximate the Prague dimension and the so-called Kneser rank of G, see [18]. In particular, we have

$$\mathrm{cc}'(\overline{G}) \le \dim_{\mathrm{P}}(G) \le \mathrm{cc}'(\overline{G}) + 1, \tag{5}$$

which follows by noting that the color classes of a properly colored collection $\mathcal{C}$ of cliques naturally correspond to subgraphs consisting of vertex-disjoint unions of cliques (the $+1$ in the upper bound is only needed to handle boundary cases where an edge is contained in cliques from all color classes); see [34, 18].

For random graphs, Füredi and Kantor [18] showed that the clique covering thickness is whp $\mathrm{cc}_\Delta(G_{n,p}) = \Theta(n/\log n)$ for constant edge-probabilities p. Supported by $\mathrm{cc}_\Delta(G) \le \mathrm{cc}'(G)$ and further evidence, they conjectured that the clique covering chromatic index is whp also $\mathrm{cc}'(G_{n,p}) = \Theta(n/\log n)$ for constant p, see [18, Conjecture 17]. The following theorem proves their chromatic index conjecture in a strong form, allowing for $p = p(n) \to 0$. More importantly, Theorem 5 and inequality (5) together imply our main Prague dimension result Theorem 1, since the complement $\overline{G_{n,p}}$ of $G_{n,p}$ has the same distribution as $G_{n,1-p}$.

Theorem 5 (Thickness and chromatic index of clique coverings of random graphs)**.** *For every fixed real $\gamma \in (0,1)$ there are constants $c > 0$ and $C = C(\gamma) > 0$ so that if the edge-probability $p = p(n)$ satisfies $n^{-1}\log n \ll p \le 1-\gamma$, then with high probability*

$$c\frac{np}{\log_{1/p} n} \le cc_\Delta(G_{n,p}) \le cc'(G_{n,p}) \le C\frac{np}{\log_{1/p} n}. \tag{6}$$

Remark 6. *Our proof shows that the upper bound in* (6) *remains valid when the definition of $cc'(G)$ is restricted to clique partitions of the edges (instead of clique coverings); see Sections 1.3.1 and 2.*

The main contribution of (6) is the upper bound, where the mild assumption $p \le 1-\gamma$ again turns out to be necessary, since Lemma 20 implies that whp $\mathrm{cc}_\Delta(G_{n,p})/(np/\log_{1/p} n) \to \infty$ as $p \to 1$. The lower bound in (6) is straightforward: it is well-known that $G_{n,p}$ whp has maximum degree $\Delta = \Theta(np)$ and largest clique of size $\omega = O(\log_{1/p} n)$, which gives $\mathrm{cc}_\Delta(G_{n,p}) \ge \Delta/(\omega-1) = \Omega(np/\log_{1/p} n)$.

1.3 Proof strategy: finding a clique partition of a random graph

We now comment on the proofs of Theorems 4–5, for which it remains to establish the upper bounds in inequalities (4) and (6). In Section 2 we shall establish these upper bounds using the following proof strategy, which finds a clique partition $\mathcal{P}$ of $G_{n,p}$ with the desired properties, i.e., size and chromatic index bounds.

Step 1: Decomposing the edges of $G_{n,p}$ into a clique partition $\mathcal{P}$. We first use a semi-random 'nibble-type' algorithm to incrementally construct a decreasing sequence of n-vertex graphs

$$G_{n,p} = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_I, \tag{7}$$

inspired by the semi-random approaches of Frieze and Reed [16] and Guo and Warnke [19]. Omitting some technicalities, the main idea is to obtain G_{i+1} from G_i by removing the edges of a random collection $\mathcal{K}_i$ of cliques of size $k_i = O(\log n)$ from G_i. We iterate this until G_I is sufficiently sparse, i.e., has maximum degree $\Delta(G_I) = o(np/(\log_{1/p} n)^2)$, say, and then put all remaining edges of G_I into $\mathcal{K}_I$, to ensure that

$$\mathcal{P} = \mathcal{K}_0 \cup \cdots \cup \mathcal{K}_I \tag{8}$$

covers all edges of $G_{n,p}$. Here we exploit the flexibility of the semi-random approach, which allows us to add extra wrinkles to the algorithm. In particular, using concentration inequalities, these extra wrinkles enable us to show that whp all graphs G_i stay pseudo-random, i.e., that G_i 'looks like' a random graph G_{n,p_i} with suitably decaying edge-probabilities p_i; see Section 2.1 and Theorem 9 for the details.

Step 2: Coloring the clique partition $\mathcal{P}$. We then use the basic observation

$$\chi'(\mathcal{P}) \le \sum_{0\le i\le I} \chi'(\mathcal{K}_i) \le \sum_{0\le i<I} \chi'(\mathcal{K}_i) + 2\Delta(G_I), \tag{9}$$

where the last inequality $\chi'(\mathcal{K}_i) \le 2\Delta(G_I)$ follows from Vizing's theorem, since $\mathcal{K}_I = E(G_I)$ simply contains all edges of G_I. Thinking of $\mathcal{K}_i$ as a hypergraph with vertex-set $V(G_i)$ and edge-set $\mathcal{K}_i$, we would like to similarly bound $\chi'(\mathcal{K}_i) = O(\Delta(\mathcal{K}_i))$, but there is a major obstacle here. Namely, as discussed, such Pippenger–Spencer type coloring results only apply to hypergraphs with edges of constant size, and their proofs are hard to extend to hypergraphs with edges of size $O(\log n)$ such as $\mathcal{K}_i$. We overcome this technical obstacle by exploiting that $\mathcal{K}_i$ is a random collection of cliques from G_i. Crucially, this enables us to bound $\chi'(\mathcal{K}_i)$ using our new probabilistic Pippenger–Spencer type result Theorem 2, which efficiently colors such random hypergraphs with large edges. In view of (3), it thus becomes plausible that whp

$$\chi'(\mathcal{P}) \;\le\; \sum_{0\le i<I} O\big(\Delta(\mathcal{K}_i)\big) \;+\; O\big(\Delta(G_I)\big), \tag{10}$$

where the pseudo-random properties of the G_i are key for verifying the technical assumptions of Theorem 2. Using again pseudo-randomness to estimate $\Delta(\mathcal{K}_i)$ and $\Delta(G_I)$, it eventually turns out[4] that whp

$$\chi'(\mathcal{P}) \;\le\; \sum_{0\le i<I} O\bigg(\frac{np_i}{\log_{1/p_i} n}\bigg) \;+\; O\big(np_I\big) \;\le\; O\bigg(\frac{np}{\log_{1/p} n}\bigg), \tag{11}$$

where the exponentially decaying edge-probabilities p_i will ensure that in estimate (11) the bulk of the contribution comes from the case $i = 0$ with $p_0 = p$; see Section 2.2.1 for the technical details. Finally, the whp size estimate $|\mathcal{P}| = O\big(n^2p/(\log_{1/p} n)^2\big)$ can be obtained in a similar but simpler way; see Section 2.2.2.

1.3.1 Technical result: weakly pseudo-random clique partition

As we shall see in Section 2, the outlined proof strategy gives the following technical result, which for large edge-probabilities $p = p(n)$ intuitively guarantees that the random graph $G_{n,p}$ has a weakly pseudo-random clique partition $\mathcal{P}$, i.e., which simultaneously has small size, thickness and chromatic index.

Theorem 7. *There is a constant $\alpha > 0$ so that, for every fixed real $\gamma \in (0, 1)$, there are constants $B, C > 0$ such that the following holds. If the edge-probability $p = p(n)$ satisfies $n^{-\alpha} \le p \le 1 - \gamma$, then whp there exists a clique partition $\mathcal{P}$ of the edges of $G_{n,p}$ satisfying $\max_{K\in\mathcal{P}} |K| \le \log_{1/p} n$, $|\mathcal{P}| \le Bn^2p/(\log_{1/p} n)^2$ and $\Delta(\mathcal{P}) \le \chi'(\mathcal{P}) \le Cnp/\log_{1/p} n$.*

Remark 8. *The proof shows that the whp conclusion in fact holds with probability at least $1 - n^{-\omega(1)}$.*

After potentially increasing the constants $B, C > 0$, this theorem readily implies the upper bounds in (4) and (6) of Theorems 4–5, since for smaller edge-probabilities $p = p(n) \le n^{-\alpha}$ the trivial clique partition $\mathcal{P} := E(G_{n,p})$ consisting of all edges of $G_{n,p}$ easily gives the desired bounds, by combining the well-known whp estimates $|\mathcal{P}| = |E(G_{n,p})| = \Theta(n^2p)$ and $\chi'(\mathcal{P}) = \chi'(G_{n,p}) \le 2\Delta(G_{n,p}) = \Theta(np)$ with $1 \le \alpha^{-1}/\log_{1/p} n$.

1.4 Organization

In Section 2 we prove our main technical clique partition result Theorem 7 (which as discussed implies Theorems 1, 4 and 5), by analyzing a semi-random greedy clique partition algorithm using concentration inequalities and our new chromatic index result Theorem 2. We then prove our key tool Theorem 2 in Section 3, by analyzing a natural random greedy edge coloring algorithm using the differential equation method. The final Section 4 discusses some open problems, sharpens the lower bounds of Theorems 4–5 for constant edge-probabilities p, and also records strengthenings of Theorems 4–5 for many small $p = p(n) \to 0$.

2 Semi-random greedy clique partition algorithm

In this section we prove Theorem 7 (and thus Theorems 1, 4 and 5, see Sections 1.2–1.3) by showing that a certain semi-random greedy algorithm is likely to find the desired clique partition $\mathcal{P}$ of the binomial random

[4] Heuristically, the form of the upper bound (11) can also be motivated as follows: (7) and $G_i \approx G_{n,p_i}$ loosely suggest $\mathrm{cc}'(G_{n,p}) \le \sum_{0\le i\le I} \mathrm{cc}'(G_{n,p_i})$, which together with (6) and $\mathrm{cc}'(G_{n,p_I}) \le 2\Delta(G_{n,p_I}) = O(np_I)$ makes the first inequality in (11) a natural target bound (the second inequality is more technical, and follows by integral comparison; see Section 2.2.1).

graph $G_{n,p}$. This algorithm iteratively adds cliques to $\mathcal{P}$, and the main idea is roughly as follows. Writing $G_i \subseteq G_{n,p}$ for the subgraph containing all edges of $G_{n,p}$ which are edge-disjoint from the cliques added to $\mathcal{P}$ during the first i iterations, we randomly sample a collection $\mathcal{K}_i$ of cliques from G_i (of suitable size k_i). We then alter this collection to ensure that there are no edge-overlaps between the cliques, and add the resulting edge-disjoint collection $\mathcal{K}_i^* \cup D_i$ of cliques to $\mathcal{P}$. Finally, after a sufficiently large number of I iterations, we add all remaining so-far uncovered edges of $G_I \subseteq G_{n,p}$ to $\mathcal{P}$ (as cliques of size two).

In fact, we shall use an additional wrinkle for technical reasons: in each iteration of the algorithm we add an extra set S_i of random edges to $\mathcal{P}$, which helps us to ensure that the graphs $G_i = ([n], E_i)$ stay pseudo-random, i.e., resemble a random graph G_{n,p_i} with suitable decaying edge-probabilities p_i.

2.1 Details of the semi-random 'nibble' algorithm

Turning to the technical details of our clique partition algorithm, let

$$k := \lceil \sigma \log_{1/p} n \rceil, \quad I := \lceil \tau k^\tau \log k \rceil, \quad p_i := p e^{-i/k^\tau}, \quad k_i := \lceil \sigma \log_{1/p_i} n \rceil, \quad \epsilon := n^{-\sigma}, \tag{12}$$

where we fix the absolute constants $\sigma := 1/9$ and $\tau := 9$ for concreteness (we have made no attempt to optimize these constants, and the reader loses little by simply assuming that σ and τ are always sufficiently small and large, respectively, whenever needed). For any vertex-subset $U \subseteq [n]$ with $|U| \le j$ we define

$$\mathcal{C}_{U,j,i} := \Big\{ J \subseteq [n] : \ U \subseteq J, \ |J| = j, \ \tbinom{J}{2} \setminus \tbinom{U}{2} \subseteq E_i \Big\}. \tag{13}$$

In words, if U forms a clique in the graph $G_i = ([n], E_i)$, then $\mathcal{C}_{U,j,i}$ corresponds to the set of all j-vertex cliques of G_i that contain U. Furthermore, if G_i indeed heuristically resembles the random graph G_{n,p_i} (as suggested above, and later made precise by Theorem 9), then we expect that $|\mathcal{C}_{U,j,i}| \approx \mu_{|U|,j,i}$, where

$$\mu_{s,j,i} := \binom{n-s}{j-s} p_i^{\binom{j}{2} - \binom{s}{2}}. \tag{14}$$

Writing $E(\mathcal{C}) := \bigcup_{K \in \mathcal{C}} E(K)$ for the edges covered by a family $\mathcal{C}$ of cliques, after defining

$$q_i := \frac{1}{(1+\epsilon) k^\tau \mu_{2,k_i,i}} \quad \text{and} \quad \zeta_{e,i} := 1 - (1-q_i)^{\max\{(1+\epsilon)\mu_{2,k_i,i} - |\mathcal{C}_{e,k_i,i}|, \, 0\}} \tag{15}$$

we now formally state the algorithm that finds the desired clique partition $\mathcal{P}$ of $G_{n,p}$.

Algorithm: Semi-random greedy clique partition

Set $\mathcal{P}_0 := \varnothing$ and $G_0 := ([n], E_0)$, where $E_0 := E(G_{n,p})$.
for $i = 0$ **to** $I - 1$ **do**
  Let $\mathcal{C}_i := \mathcal{C}_{\varnothing, k_i, i}$ contain all k_i-vertex cliques of G_i.
  Generate $\mathcal{K}_i \subseteq \mathcal{C}_i$: independently include each clique $K \in \mathcal{C}_i$ with probability q_i.
  Generate $S_i \subseteq E_i$: independently include each edge $e \in E_i$ with probability $\zeta_{e,i}$.
  Let $\mathcal{K}_i^*$ be a size-maximal collection of edge-disjoint k_i-vertex cliques in $\mathcal{K}_i$.
  Set $\mathcal{P}_{i+1} := \mathcal{P}_i \cup \mathcal{K}_i^* \cup D_i \cup \big(S_i \setminus E(\mathcal{K}_i)\big)$, where $D_i := E(\mathcal{K}_i) \setminus E(\mathcal{K}_i^*)$.
  Set $G_{i+1} := ([n], E_{i+1})$, where $E_{i+1} := E_i \setminus \big(E(\mathcal{K}_i) \cup S_i\big)$.
end for
Return $\mathcal{P} := \mathcal{P}_I \cup E_I$.

One may heuristically motivate the technical definitions (15) of q_i and $\zeta_{e,i}$ as follows. The 'inclusion' probability q_i will intuitively ensure that, for any fixed edge $e \in E_i$, the expected number of cliques in $\mathcal{K}_i$ containing e is roughly $|\mathcal{C}_{e,k_i,i}| \cdot q_i \approx \mu_{2,k_i,i} \cdot q_i \approx 1/k^\tau$. This makes it plausible that the cliques in $\mathcal{K}_i$ are largely edge-disjoint, i.e., that $|\mathcal{K}_i^*| \approx |\mathcal{K}_i|$. The 'stabilization' probability $\zeta_{e,i}$ will intuitively ensure that

$$\mathbb{P}(e \in E_{i+1} \mid e \in E_i) = (1-q_i)^{|\mathcal{C}_{e,k_i,i}|} \cdot (1 - \zeta_{e,i}) \approx (1-q_i)^{(1+\epsilon)\mu_{2,k_i,i}} \approx e^{-1/k^\tau}.$$

Since all edges $e \in E_i$ of G_i have roughly the same probability of appearing in E_{i+1}, it then inductively becomes plausible that G_{i+1} resembles a random graph $G_{n,p_{i+1}}$ with edge-probability $p_{i+1} \approx p_i \cdot e^{-1/k^\tau}$.

2.2 The clique partition $\mathcal{P}$: proof of Theorem 7

In this section we prove Theorem 7 by analyzing the clique partition $\mathcal{P}$ produced by the semi-random greedy algorithm. Recalling the definitions (13)–(14) of $|\mathcal{C}_{U,j,i}|$ and $\mu_{s,j,i}$, Theorem 9 confirms our heuristic that G_i stays pseudo-random, i.e., resembles the random graph G_{n,p_i} with respect to various clique statistics.

Theorem 9 (Pseudo-randomness of the graphs G_i)**.** *Let $p = p(n)$ satisfy $n^{-\sigma/\tau} \le p \le 1-\gamma$, where $\gamma \in (0,1)$ is a constant. Then, with probability at least $1 - n^{-\omega(1)}$, for all $0 \le i \le I$ the following event $\mathcal{R}_i$ holds: for all $U \subseteq [n]$ and j with $0 \le |U| \le j \le k_i$, we have*

$$|\mathcal{C}_{U,j,i}| \;=\; (1 \pm \epsilon) \cdot \mu_{|U|,j,i}. \tag{16}$$

We defer the proof of this important technical auxiliary result to Section 2.3, and first use it (together with our new edge-coloring result Theorem 2) to prove Theorem 7 with $\alpha := \sigma/\tau$. To this end, we henceforth tacitly assume $n^{-\sigma/\tau} \le p \le 1-\gamma$. In particular, for $0 \le i \le I$ it then is routine[5] to check that

$$8 < \tau - o(1) \le \frac{\sigma \log n}{\log(k^{2\tau}/p)} \le k_i \le k = n^{o(1)} \qquad \text{and} \qquad \min_{0 \le s \le k_i - 1} p_i^s \ge p_i^{k_i-1} \ge n^{-\sigma}. \tag{17}$$

By construction of $\mathcal{P}$, it also follows that $\max_{K \in \mathcal{P}} |K| \le k \le \log_{1/p} n$. To complete the proof of Theorem 7, it thus remains to bound the size and chromatic index of the clique partition $\mathcal{P}$.

2.2.1 Chromatic index of $\mathcal{P}$

We first focus on the chromatic index of the clique partition $\mathcal{P}$, which is easily seen to be (by separately coloring different subsets of the cliques, using disjoint sets of colors) at most

$$\chi'(\mathcal{P}) \le \sum_{0 \le i \le I-1} \Big(\chi'(\mathcal{K}_i) + \chi'(D_i) + \chi'(S_i) \Big) + \chi'(E_I). \tag{18}$$

For $\mathcal{S} \in \{\mathcal{K}_i, D_i, S_i, E_I\}$, let $\mathcal{S}^{(v)} \subseteq \mathcal{S}$ denote the subset of cliques that contain the vertex v. Since the cliques in D_i, S_i, E_I are all simply edges, using Vizing's theorem it follows that

$$\chi'(\mathcal{P}) \le \sum_{0 \le i \le I-1} \Big(\chi'(\mathcal{K}_i) + \max_{v \in [n]} \big|D_i^{(v)}\big| + \max_{v \in [n]} \big|S_i^{(v)}\big| \Big) + \max_{v \in [n]} \big|E_I^{(v)}\big| + (2I+1). \tag{19}$$

In the following we bound the contributions of each of these terms. We start with the main term $\chi'(\mathcal{K}_i)$, where the trivial upper bound $\chi'(\mathcal{K}_i) \le O(k_i) \cdot \Delta(\mathcal{K}_i)$ would be too weak for our purposes. Gearing up to instead apply our stronger Pippenger–Spencer type chromatic index result Theorem 2 to the random set $\mathcal{K}_i \subseteq \mathcal{C}_i$ of cliques, let $\mathcal{H} := ([n], \mathcal{C}_i)$ denote the k_i-uniform auxiliary hypergraph consisting of all k_i-vertex cliques in G_i. Note that $\mathcal{K}_i$ has the same distribution as the edge-set of the random subhypergraph $\mathcal{H}_{q_i}$, where each edge $e \in \mathcal{H}$ is independently included with probability q_i. This differs slightly from the random subhypergraph $\mathcal{H}_m$ considered in Theorem 2, which consists of m random edges that are independently chosen from $\mathcal{H}$. As usual, this minor difference is negligible for our purposes, since standard arguments from random graph theory show that $\chi'(\mathcal{H}_{q_i})$ behaves similarly to $\chi'(\mathcal{H}_m)$ when the expected number of edges in $\mathcal{H}_{q_i}$ is close to m, as formalized by the following corollary of Theorem 2 that applies to $\mathcal{H}_{q_i}$ with $r = k_i$ and $q = q_i$ (the proof of Corollary 10 is deferred to Appendix B, since the routine details of this standard reduction are rather tangential to the main argument here).

Corollary 10 (Convenient variant of Theorem 2)**.** *There is $\xi = \xi(\delta) > 0$ such that if the assumptions of Theorem 2 hold for a given n-vertex r-uniform hypergraph $\mathcal{H}$, with assumption $m \le n^{rn^{\sigma/5}}$ replaced by $m \le \xi e(\mathcal{H})$, then we have $\mathbb{P}(\chi'(\mathcal{H}_q) \le (1+2\delta)rm/n) \ge 1 - n^{-\omega(r)}$, where $\mathcal{H}_q$ denotes the random subhypergraph of $\mathcal{H}$ where each edge $e \in \mathcal{H}$ is independently included with probability $q := m/|E(\mathcal{H})|$.*

[5]To see the claimed bounds in (17), note that $1/p_i \le 1/p_I \le k^{2\tau}/p \le n^{\sigma/\tau + o(1)}$ and $p_i^{k_i-1} \ge p_i^{\sigma \log_{1/p_i} n} = n^{-\sigma}$.

Conditional on $\mathcal{R}_i$, we will apply Corollary 10 to $\mathcal{H} = ([n], \mathcal{C}_i)$ with $r := k_i$, $m := |E(\mathcal{H})|q_i$, $D := \mu_{1,k_i,i}$, $q := q_i$, as well as

$$b := 2\sigma/\log(1/(1-\gamma)) \quad \text{and} \quad \delta := 30b/\sigma. \tag{20}$$

We now verify the technical assumptions of Corollary 10 (and thus Theorem 2). Using the definition (12) of k_i and inequality (17) together with $p_i \le p \le 1-\gamma$, we obtain $2 < k_i \le 2\sigma(\log n)/\log(1/p_i) \le b \log n$. Using the estimate (16) of $\mathcal{R}_i$ together with the definition (15) of q_i and the definition (14) of $\mu_{s,j,i}$, it follows that

$$m = |\mathcal{C}_{\varnothing,k_i,i}| \cdot q_i = \frac{(1\pm\epsilon)\mu_{0,k_i,i}}{(1+\epsilon)k^\tau \mu_{2,k_i,i}} = \frac{1\pm\epsilon}{1+\epsilon} \cdot \frac{\binom{n}{k_i} p_i^{\binom{k_i}{2}}}{k^\tau \binom{n}{k_i-2} p_i^{\binom{k_i}{2}-1}} = \frac{1\pm\epsilon}{1+\epsilon} \cdot \frac{n(n-1)p_i}{k_i(k_i-1)k^\tau}, \tag{21}$$

so that $m \ge n^{2-\sigma-o(1)} \gg n^{1+\sigma}$ by (17) and choice of σ. Recalling that $\epsilon = n^{-\sigma}$, estimate (16) implies that $\mathcal{H} = ([n], \mathcal{C}_i)$ satisfies the degree condition in (2). We also have $\mu_{2,k_i,i}/\mu_{1,k_i,i} \le \big(\Omega(n/k_i) \cdot p_i\big)^{-1} \le n^{-1+\sigma+o(1)} \ll n^\sigma$, which in view of (16) and $D = \mu_{1,k_i,i}$ implies that $\mathcal{H}$ also satisfies the codegree condition in (2). We similarly infer $D = \big(\Omega(n/k_i) \cdot p_i^{k_i/2}\big)^{k_i-1} \ge (n^{1-\sigma-o(1)})^4 \gg n^3$, so that $m = O(n^2/k_i) \ll D/r \ll e(\mathcal{H})$. We thus may apply Corollary 10 to $\mathcal{H}$, which together with our above discussion gives

$$\mathbb{P}\big(\chi'(\mathcal{K}_i) \ge (1+2\delta)k_i m/n \mid \mathcal{R}_i\big) = \mathbb{P}\big(\chi'(\mathcal{H}_q) \ge (1+2\delta)k_i m/n \mid \mathcal{R}_i\big) \le n^{-\omega(1)}. \tag{22}$$

In the following we fix a vertex $v \in [n]$, and bound $|S_i^{(v)}|$ and $|D_i^{(v)}|$ separately. For these terms we will have some elbow-room, and we can thus be more generous in our upcoming estimates. Using estimates (16)–(17) together with Bernoulli's inequalty and the definitions (14)–(15) of $\mu_{s,j,i}$ and q_i, it follows that $1-(1-q_i)^{2\epsilon\mu_{2,k_i,i}} \le 2\epsilon\mu_{2,k_i,i}q_i \le 2\epsilon k^{-\tau}$ and $\epsilon = n^{-\sigma} \ll k^{-2} \le k_i^{-2}$ as well as

$$\begin{aligned}\mathbb{E}\big(|S_i^{(v)}| \mid \mathcal{R}_i\big) &= \sum_{e \in \mathcal{C}_{\{v\},2,i}} \zeta_{e,i} \le |\mathcal{C}_{\{v\},2,i}| \cdot \big(1-(1-q_i)^{2\epsilon\mu_{2,k_i,i}}\big) \\ &\le (1+\epsilon)\mu_{1,2,i} \cdot 2\epsilon k^{-\tau} \le 2np_i \cdot 2\epsilon k^{-\tau} \ll \frac{np_i}{k_i^2 k^\tau} =: \lambda.\end{aligned} \tag{23}$$

Note that $\lambda \ge n^{1-\sigma-o(1)} \gg \log n$ by inequality (17) and choice of σ. Furthermore, since $|S_i^{(v)}|$ is a sum of independent indicator random variables, standard Chernoff bounds (such as [21, Theorem 2.1]) imply

$$\mathbb{P}\big(|S_i^{(v)}| \ge 2\lambda \mid \mathcal{R}_i\big) \le \exp\big(-\Theta(\lambda)\big) \le n^{-\omega(1)}. \tag{24}$$

Turning to $|D_i^{(v)}|$, let X denote the number of unordered pairs $\{K', K''\} \in \binom{\mathcal{K}_i}{2}$ with $|\{K', K''\} \cap \mathcal{K}_i^{(v)}| \ge 1$ and $|E(K') \cap E(K'')| \ge 1$. Since each of these edge-overlapping clique pairs contributes at most $k_i \le k$ edges to $|D_i^{(v)}|$, we infer $|D_i^{(v)}| \le kX$. Furthermore, using (16) and (15), it follows similarly to (21) that

$$\begin{aligned}\mathbb{E}(X \mid \mathcal{R}_i) &\le \sum_{K' \in \mathcal{C}_{\{v\},k_i,i}} \sum_{e \in \binom{K'}{2}} \sum_{K'' \in \mathcal{C}_{e,k_i,i}} q_i^2 \le (1+\epsilon)\mu_{1,k_i,i} \cdot \binom{k_i}{2} \cdot (1+\epsilon)\mu_{2,k_i,i} \cdot q_i^2 \\ &= \frac{\binom{k_i}{2}\mu_{1,k_i,i}}{k^{2\tau}\mu_{2,k_i,i}} = \frac{\binom{k_i}{2}\binom{n-1}{k_i-1}p_i^{\binom{k_i}{2}}}{k^{2\tau}\binom{n-2}{k_i-2}p^{\binom{k_i}{2}-1}} \le \frac{k_i n p_i}{k^{2\tau}} =: \mu.\end{aligned} \tag{25}$$

Conditioning on the event $\mathcal{R}_i$, we shall bound X using the following upper tail inequality for combinatorial random variables, which is a convenient corollary of [44, Theorem 9]. Note that $X = |\mathcal{I}^+|$ in Lemma 11.

Lemma 11. *Let $(\xi_j)_{j\in\Lambda}$ be a finite family of independent random variables with $\xi_j \in \{0,1\}$. Let $\mathcal{I} \subseteq 2^\Lambda$ be a finite family of subsets of Λ. Let $(Y_\alpha)_{\alpha\in\mathcal{I}}$ be a family of random variables with $Y_\alpha := \mathbb{1}_{\{\xi_j=1 \text{ for all } j\in\alpha\}}$. Defining $\mathcal{I}^+ := \{\alpha \in \mathcal{I} : Y_\alpha = 1\}$, let $\mathcal{G}$ be an event that implies $\max_{\alpha\in\mathcal{I}^+} |\{\beta \in \mathcal{I}^+ : \beta \cap \alpha \ne \varnothing\}| \le C$. Set $X := \sum_{\alpha\in\mathcal{I}} Y_\alpha$, and assume that $\mathbb{E}\, X \le \mu$. Then, for all $x > \mu$,*

$$\mathbb{P}(X \ge x \text{ and } \mathcal{G}) \;\le\; \big(e\mu/x\big)^{x/C}. \tag{26}$$

We will apply Lemma 11 to X with $\Lambda = \mathcal{C}_i$, the independent random variables $\xi_K := \mathbb{1}_{\{K \in \mathcal{K}_i\}}$, and $\mathcal{I}$ equal to the set of unordered pairs $\{K', K''\} \in \binom{\mathcal{C}_i}{2}$ with $|\{K', K''\} \cap \mathcal{C}_{\{v\},k_i,i}| \geq 1$ and $|E(K') \cap E(K'')| \geq 1$. Let $\mathcal{G}$ denote that the event that each edge $e \in E_i$ is contained in at most $z := \lceil \log n \rceil$ cliques in $\mathcal{K}_i$. Clearly, $\mathcal{G}$ implies that each clique $K' \in \mathcal{K}_i$ has edge-overlaps with a total of at most $\binom{k_i}{2} \cdot z$ cliques $K'' \in \mathcal{K}_i$, so that the parameter $C := 2 \cdot \binom{k_i}{2} z \leq k^2 z$ works in Lemma 11. Recalling $|D_i^{(v)}| \leq kX$, by invoking inequality (26) with $x := \lambda/k \geq k^{\tau-4}\mu > e^2\mu$ it follows that

$$\mathbb{P}\big(|D_i^{(v)}| \geq \lambda \text{ and } \mathcal{G} \mid \mathcal{R}_i\big) \leq \mathbb{P}\big(X \geq \lambda/k \text{ and } \mathcal{G} \mid \mathcal{R}_i\big) \leq \big(e\mu/x\big)^{x/C} \leq \exp\big(-\Theta(\lambda/(k^3 z))\big) \leq n^{-\omega(1)}, \quad (27)$$

where the last inequality uses $\lambda/(k^3 z) \geq \lambda n^{-o(1)} \gg \log n$ analogous to (24). With an eye on the event $\mathcal{G}$, note that conditional on $\mathcal{R}_i$ we have $|\mathcal{C}_{e,k_i,i}| q_i \leq k^{-\tau} \leq 1$ for each edge $e \in E_i$. Together with $z = \lceil \log n \rceil$, by taking a union bound over all edges $e \in E_i$ it now routinely follows that

$$\mathbb{P}(\neg\mathcal{G} \mid \mathcal{R}_i) \leq \sum_{e \in E_i} \binom{|\mathcal{C}_{e,k_i,i}|}{z} q_i^z \leq |E_i| \cdot \Big(|\mathcal{C}_{e,k_i,i}| q_i e/z\Big)^z \leq n^2 \cdot (e/z)^z \leq n^{-\omega(1)}. \quad (28)$$

Finally, with the above inequalities in hand, we are now ready to further estimate the upper bound (19) on the chromatic index of $\mathcal{P}$. Indeed, by combining the above inequalities for $\chi'(\mathcal{K}_i)$, $|S_i^{(v)}|$ and $|D_i^{(v)}|$ from (22), (24), and (27)–(28) with the degree estimate $|E_I^{(v)}| = |\mathcal{C}_{\{v\},k_I,I}| = (1 \pm \epsilon)(n-1)p_I$ from (16) and $I = n^{o(1)}$, using inequality (19) and Theorem 9 it follows by a standard union bound argument (which enables us to bound the contributions of each term in (19) separately) that the chromatic index of $\mathcal{P}$ is whp[6] at most

$$\chi'(\mathcal{P}) \leq \sum_{0 \leq i \leq I-1} \left(\frac{(1+2\delta)2np_i}{k_i k^\tau} + \frac{2np_i}{k_i^2 k^\tau} + \frac{np_i}{k_i^2 k^\tau} \right) + (1+\epsilon)np_I + n^{o(1)}, \quad (29)$$

where the $k_i^2 > k_i$ term will be useful in Section 2.2.2. Let $\pi := \log(1/p)$ and $f(x) := e^{-x}(1 + x/\pi)$. Using $p_i = p \cdot e^{-i/k^\tau}$ and $k_i \geq \sigma \log_{1/p}(n)/(1 + i/(k^\tau \pi))$ as well as $n^\sigma \ll n^{1-\sigma} \leq np_I \leq np/k^\tau$, it follows that

$$\chi'(\mathcal{P}) \leq \frac{(5+4\delta)np}{\sigma \log_{1/p} n} \sum_{0 \leq i \leq I-1} \frac{f(i/k^\tau)}{k^\tau} + \frac{3np}{\big(\sigma \log_{1/p} n\big) k^{\tau-1}}. \quad (30)$$

On $[0, \infty)$ the function $f(x)$ first increases and then decreases, with a maximum at $x^* := \max\{0, 1-\pi\}$. By comparing the sum with an integral, it then is standard to see that

$$\sum_{0 \leq i \leq I-1} \frac{f(i/k^\tau)}{k^\tau} \leq \int_0^\infty f(x)dx + 2f(x^*)/k^\tau \ = \ 1 + \pi^{-1} + 2f(x^*)/k^\tau \ \leq \ 1 + O\big(\pi^{-1} + k^{-\tau}\big). \quad (31)$$

Combining inequalities (30)–(31) with the definition (20) of δ, after noting $\pi \geq \log(1/(1-\gamma)) > 0$ and $\min\{k^\tau, k^{\tau-1}\} > 1$ it follows that there is a constant $C = C(\sigma, \gamma) > 0$ such that whp $\chi'(\mathcal{P}) \leq Cnp/\log_{1/p} n$.

2.2.2 Size of $\mathcal{P}$

It remains to bound the size of the clique partition $\mathcal{P}$, which by construction is at most

$$|\mathcal{P}| \leq \sum_{0 \leq i \leq I-1} \Big(|\mathcal{K}_i| + |D_i| + |S_i|\Big) + |E_I|. \quad (32)$$

Rather than estimating each of these terms (which is conceptually straightforward), we shall instead reuse known estimates from Section 2.2.1. Recall that $\mathcal{K}_i \subseteq \mathcal{C}_i$ contains cliques $K \in \mathcal{K}_i$ with $|K| = k_i$ vertices. A routine double-counting argument thus gives $|\mathcal{K}_i| \cdot k_i \leq \sum_{K \in \mathcal{K}_i} |K| = \sum_{v \in [n]} |\mathcal{K}_i^{(v)}| \leq n \cdot \chi'(\mathcal{K}_i)$. Recalling that D_i, S_i, E_I are simply sets of edges, it follows that

$$|\mathcal{P}| \leq \sum_{0 \leq i \leq I-1} \Big(n/k_i \cdot \chi'(\mathcal{K}_i) + n \cdot \max_{v \in [n]} \big|D_i^{(v)}\big| + n \cdot \max_{v \in [n]} \big|S_i^{(v)}\big|\Big) + n \cdot \max_{v \in [n]} \big|E_I^{(v)}\big|. \quad (33)$$

[6] In fact, inequality (29) holds with probability at least $1 - n^{-\omega(1)}$, since each of the $O(I \cdot n) = n^{O(1)}$ many events considered fails with probability at most $n^{-\omega(1)}$. A similar remark applies to inequality (34) in Section 2.2.2.

After comparing the above upper bound (33) for $|\mathcal{P}|$ with (19), we see that the proof of (29) implies the following estimate: the size of $\mathcal{P}$ is whp at most

$$|\mathcal{P}| \le \sum_{0\le i\le I-1}\left(\frac{(1+2\delta)2n^2p_i}{k_i^2k^\tau}+\frac{2n^2p_i}{k_i^2k^\tau}+\frac{n^2p_i}{k_i^2k^\tau}\right) \;+\; (1+\epsilon)n^2p_I. \tag{34}$$

Recalling $\pi = \log(1/p)$, set $g(x) := e^{-x}(1+x/\pi)^2$. Proceeding similarly to (29)–(31), using $\int_0^\infty g(x)dx = 1+O(\pi^{-1}+\pi^{-2})$ it follows that there is a constant $B = B(\sigma,\gamma) > 0$ such that whp

$$|\mathcal{P}| \le \frac{(5+4\delta)n^2p}{\left(\sigma\log_{1/p} n\right)^2}\sum_{0\le i\le I-1}\frac{g(i/k^\tau)}{k^\tau} \;+\; \frac{2n^2p}{\left(\sigma\log_{1/p} n\right)^2k^{\tau-2}} \;\le\; B\frac{n^2p}{(\log_{1/p} n)^2}, \tag{35}$$

which completes the proof Theorem 7 (modulo the deferred proof of Theorem 9). □

2.3 Pseudo-randomness of the graphs G_i: proof of Theorem 9

In this section we give the deferred proof of Theorem 9. For technical reasons, we will establish concentration of the $|\mathcal{C}_{U,j,i}|$ variables in a somewhat indirect way, by focusing on auxiliary random variables that are more amenable to concentration inequalities. Turning to the details, for any vertex-subset $U \subseteq [n]$ we define

$$N_{U,i} := \bigl|\bigl\{w \in [n]\setminus U \,:\, U\times\{w\}\subseteq E_i\bigr\}\bigr|. \tag{36}$$

In words, $N_{U,i}$ denotes the number of common neighbors of U in $G_i = ([n], E_i)$. Recalling that G_i heuristically resembles the random graph G_{n,p_i}, we expect that $N_{U,i} \approx (n-|U|)p_i^{|U|}$; so to avoid clutter we set

$$\lambda_{s,i} := (n-s)p_i^s. \tag{37}$$

The following pseudo-random result establishes Theorem 9 by confirming this heuristic prediction. Our proof of Theorem 12 exploits the technical definition of $E_{i+1} = E_i \setminus \bigl(E(\mathcal{K}_i)\cup S_i\bigr)$: the extra 'stabilization' set S_i will intuitively ensure that edges of E_i remain in E_{i+1} with roughly the correct probability, see (44)–(45).

Theorem 12 (Strengthening of Theorem 9)**.** *Let $p = p(n)$ satisfy $n^{-\sigma/\tau} \le p \le 1-\gamma$, where $\gamma \in (0,1)$ is a constant. Then, with probability at least $1-n^{-\omega(1)}$, for all $0 \le i \le I$ the following event $\mathcal{N}_i$ holds: for all $U \subseteq [n]$ with $0 \le |U| \le k_i - 1$, we have*

$$N_{U,i} \;=\; \bigl(1 \pm (i+1)\epsilon^2\bigr)\cdot\lambda_{|U|,i}. \tag{38}$$

Furthermore, $\mathcal{N}_i$ implies the event $\mathcal{R}_i$ from Theorem 9 for $0 \le i \le I$ and $n \ge n_0(\sigma,\tau)$.

Proof. Noting $kI\epsilon^2 \le n^{o(1)-\sigma}\epsilon \ll \epsilon$ it is routine to see that $\mathcal{N}_i$ implies $\mathcal{R}_i$, but we include the proof for completeness. Fixing $U \subseteq [n]$, $0 \le i \le I$ and j with $|U| \le j \le k_i$, we shall double-count the number of vertex-sequences $x_{|U|+1},\dots,x_j \in [n]\setminus U$ with the property that $U\cup\{x_{|U|+1},\dots,x_j\} \in \mathcal{C}_{U,j,i}$. Using (38) to sequentially estimate the number of common neighbors of $U\cup\{x_{|U|+1},\dots,x_s\}$, noting $j\cdot I\epsilon^2 \le kI\epsilon^2 \ll \epsilon$ it follows that

$$(j-|U|)!\cdot|\mathcal{C}_{U,j,i}| = \prod_{|U|\le s\le j-1}\Bigl(\bigl(1+O(I\epsilon^2)\bigr)\cdot(n-s)p_i^s\Bigr) = (1+o(\epsilon))\cdot\mu_{|U|,j,i}\cdot(j-|U|)!,$$

which readily gives (16) for $n \ge n_0(\sigma,\tau)$, establishing the claim that $\mathcal{N}_i$ implies $\mathcal{R}_i$. With this implication and $I = n^{o(1)}$ in mind, the below auxiliary Lemmas 13–14 then complete the proof of Theorem 12. □

Lemma 13. *We have $\mathbb{P}(\neg\mathcal{N}_0) \le n^{-\omega(1)}$.*

Lemma 14. *We have $\mathbb{P}(\neg\mathcal{N}_{i+1} \mid \mathcal{N}_i) \le n^{-\omega(1)}$ for all $0 \le i < I$.*

Proof of Lemma 13. Fix $U \subseteq [n]$ with $|U| \le k-1$, where $k = k_0$. Note that $N_{U,0}$ has a Binomial distribution with $\mathbb{E}\,N_{U,0} = (n-|U|)p^{|U|} = \lambda_{|U|,0}$, where $p = p_0$. Since $\epsilon^4\lambda_{|U|,0} = \Theta\bigl(n^{1-4\sigma}p_0^{|U|}\bigr) \ge \Omega(n^{1-5\sigma}) \gg k\log n$ by inequality (17) and choice of σ, standard Chernoff bounds (such as [21, Theorem 2.1]) imply that

$$\mathbb{P}\bigl(|N_{U,0}-\lambda_{|U|,0}| \ge \epsilon^2\lambda_{|U|,0}|\bigr) \le 2\cdot\exp\bigl(-\Theta\left(\epsilon^4\lambda_{|U|,0}\right)\bigr) \le n^{-\omega(k)}, \tag{39}$$

which completes the proof by taking a union bound over all $n^{O(k)}$ choices of the sets U. □

Conditioning on the event $\mathcal{N}_i$, in the proof of Lemma 14 we shall estimate $N_{U,i+1}$ using the following bounded differences inequality for Bernoulli variables, see [43, Corollary 1.4] and [31, Theorem 3.8].

Lemma 15. *Let $(\xi_\alpha)_{\alpha\in\mathcal{I}}$ be a finite family of independent random variables with $\xi_\alpha \in \{0,1\}$. Let $f : \{0,1\}^{|\mathcal{I}|} \to \mathbb{R}$ be a function, and assume that there exist numbers $(c_\alpha)_{\alpha\in\mathcal{I}}$ such that the following holds for all $z = (z_\alpha)_{\alpha\in\mathcal{I}} \in \{0,1\}^{|\mathcal{I}|}$ and $z' = (z'_\alpha)_{\alpha\in\mathcal{I}} \in \{0,1\}^{|\mathcal{I}|}$: $|f(z) - f(z')| \le c_\beta$ if $z_\alpha = z'_\alpha$ for all $\alpha \neq \beta$. Define $X := f\big((\xi_\alpha)_{\alpha\in\mathcal{I}}\big)$, $V := \sum_{\alpha\in\mathcal{I}} c_\alpha^2\, \mathbb{P}(\xi_\alpha = 1)$, and $C := \max_{\alpha\in\mathcal{I}} c_\alpha$. Then, for all $t \ge 0$,*

$$\mathbb{P}(|X - \mathbb{E}\, X| \ge t) \;\le\; 2\cdot \exp\left(-\frac{t^2}{2(V+Ct)}\right). \tag{40}$$

Proof of Lemma 14. To avoid clutter, we henceforth omit the conditioning on $\mathcal{N}_i$ from our notation. Fix $U \subseteq [n]$ with $|U| \le k_i - 1$. Gearing up to apply Lemma 15 to $N_{U,i+1}$, note that the associated parameter V is given by

$$V = \sum_{K\in\mathcal{C}_i} c_K^2 \cdot q_i + \sum_{e\in E_i} \hat{c}_e^2 \cdot \zeta_{e,i}, \tag{41}$$

where c_K is an upper bound on how much $N_{U,i+1}$ can change if we alter whether the clique K is in $\mathcal{K}_i$ or not, and $\hat{c}_e$ is an upper bound on how much $N_{U,i+1}$ can change if we alter whether the edge e is in S_i or not. To estimate c_K and $\hat{c}_e$, note that any edge in $U \times \{w\}$ uniquely determines w. By definition (36) of $N_{U,i+1}$, it follows that $\hat{c}_e \le 1$ and $c_K \le \binom{k_i}{2} \le k^2$, say. In addition, the number of edges $e \in E_i$ with $\hat{c}_e \neq 0$ is at most $N_{U,i} \cdot |U|$. Similarly, the number of cliques $K \in \mathcal{C}_i$ with $c_K \neq 0$ is at most $N_{U,i} \cdot |U| \cdot \max_e |\mathcal{C}_{e,k_i,i}| \le N_{U,i}|U| \cdot (1+\epsilon)\mu_{2,k_i,i}$, where we used that $\mathcal{N}_i$ implies $\mathcal{R}_i$ (as established above) to bound $|\mathcal{C}_{e,k_i,i}|$ via (16). Since $(1+\epsilon)\mu_{2,k_i,i}\cdot q_i = k^{-\tau}$ by definition (15) of q_i, and $\zeta_{e,i} \ll k^{-\tau}$ by the calculation above (23), using $|U| \le k$ and $\tau \ge 5$ we infer that

$$V \le N_{U,i}|U| \cdot k^{-\tau} \cdot k^4 + N_{U,i}|U| \cdot k^{-\tau} \le 2N_{U,i} \le 4\lambda_{|U|,i} = \Theta(\lambda_{|U|,i+1}),$$

where we used (38) and $i\epsilon^2 \le I\epsilon^2 \le n^{o(1)-2\sigma} \ll 1$ to bound $N_{U,i}$. Invoking inequality (40) of Lemma 15 with $C = k^2$, noting $C\epsilon^2 \le n^{o(1)-2\sigma} \ll 1$ it follows that

$$\mathbb{P}\big(|N_{U,i+1} - \mathbb{E}\, N_{U,i+1}| \ge 0.5\epsilon^2\lambda_{|U|,i+1}\big) \le 2\cdot\exp\big(-\Theta(\epsilon^4\lambda_{|U|,i+1})\big) \le n^{-\omega(k)},$$

where the last estimate is analogous to (39). To complete the proof it thus suffices to show that

$$\big|\mathbb{E}\, N_{U,i+1} - \lambda_{|U|,i+1}\big| \le (i+1.5)\epsilon^2\lambda_{|U|,i+1}. \tag{42}$$

Indeed, $\mathbb{P}(\neg\mathcal{N}_{i+1} \mid \mathcal{N}_i) \le n^{-\omega(1)}$ then follows by taking a union bound over all $n^{O(k)}$ sets U.

Turning to the remaining proof of (42), note that by construction

$$\mathbb{E}\, N_{U,i+1} = \sum_{\substack{w\in V\setminus U:\\ U\times\{w\}\subseteq E_i}} \mathbb{P}\big(U \times \{w\} \subseteq E_{i+1}\big). \tag{43}$$

In the following estimates (44)–(45), the technical definition (15) of the stabilization probability $\zeta_{e,i}$ will play a key role in showing that the probabilities $\mathbb{P}\big(U \times \{w\} \subseteq E_{i+1}\big)$ are similar for all relevant U and w. To this end, let us henceforth tacitly assume $U \times \{w\} \subseteq E_i$. Since $\mathcal{N}_i$ implies $\mathcal{R}_i$ we obtain $(1+\epsilon)\mu_{2,k_i,i} \ge |\mathcal{C}_{e,k_i,i}|$ via (16), so recalling $E_{i+1} = E_i \setminus \big(E(\mathcal{K}_i) \cup S_i\big)$ and the definition (15) of $\zeta_{e,i}$ it follows that

$$\begin{aligned}\mathbb{P}\left(U \times \{w\} \subseteq E_{i+1}\right) &= (1-q_i)^{|\bigcup_{e\in U\times\{w\}} \mathcal{C}_{e,k_i,i}|} \cdot \prod_{e\in U\times\{w\}} (1-\zeta_{e,i}) \\ &= (1-q_i)^{|\bigcup_{e\in U\times\{w\}} \mathcal{C}_{e,k_i,i}| - \sum_{e\in U\times\{w\}} |\mathcal{C}_{e,k_i,i}|} \cdot (1-q_i)^{|U|(1+\epsilon)\mu_{2,k_i,i}}.\end{aligned} \tag{44}$$

Recalling the definition (15) of q_i, using estimates (16)–(17) we infer that

$$\begin{aligned} q_i \cdot \left|\Big|\bigcup_{e\in U\times\{w\}} \mathcal{C}_{e,k_i,i}\Big| - \sum_{e\in U\times\{w\}} |\mathcal{C}_{e,k_i,i}|\right| &\le q_i \sum_{u\neq v\in U} |\mathcal{C}_{\{u,v,w\},k_i,i}| \le k^2\mu_{3,k_i,i}/\mu_{2,k_i,i} \\ &\le k^2/\big(\Omega(n/k_i)p_i^2\big) \le n^{-1+\sigma+o(1)} \ll n^{-2\sigma} = \epsilon^2. \end{aligned}$$

We similarly obtain $q_i \ll \epsilon^2$ and $q_i \cdot |U|(1+\epsilon)\mu_{2,k_i,i} = |U|/k^\tau \le 1$. Inserting $1 - q_i = e^{-(1+O(q_i))q_i}$ into (44), using $e^{o(\epsilon^2)} = 1 + o(\epsilon^2)$ it routinely follows that

$$\mathbb{P}\left(U \times \{w\} \subseteq E_{i+1}\right) = \left(1 + o(\epsilon^2)\right) \cdot e^{-|U|/k^\tau}. \tag{45}$$

Recalling (43) and $(i+1)\epsilon^2 \le I\epsilon^2 \ll \epsilon \ll 1$, using (38) and $\lambda_{|U|,i} \cdot e^{-|U|/k^\tau} = \lambda_{|U|,i+1}$ we infer that

$$\mathbb{E}\, N_{U,i+1} = N_{U,i} \cdot \left(1 + o(\epsilon^2)\right) e^{-|U|/k^\tau} = \Big(1 \pm \big(i + 1 + o(1)\big)\epsilon^2\Big) \cdot \lambda_{|U|,i+1},$$

which establishes (42) with room to spare, completing the proof of Lemma 14. □

For the interested reader we remark that the arguments of this section, and thus the proof of Theorem 7, carry over to any (random or deterministic) initial graph G_0 for which Lemma 13 remains true.

3 Random greedy edge coloring algorithm

In this section we prove Theorem 2 by showing that the following simple random greedy algorithm is likely to produce the desired proper edge coloring of the random edges from the hypergraph $\mathcal{H}$ (allowing for repeated edges), using the colors $[q] = \{1, \ldots, q\}$ for suitable $q \ge 1$. For $i \ge 0$, we sequentially choose an edge $e_{i+1} \in E(\mathcal{H})$ uniformly at random, and then assign e_{i+1} a color c chosen uniformly at random from all colors in $[q]$ that are still available at e_{i+1}, i.e., which have not been assigned to an edge e_j with $e_j \cap e_{i+1} \neq \varnothing$ and $j \le i$ (this also ensures the usage of different colors for each occurrence of the same edge). This random greedy coloring algorithm terminates when no more colors are available at some edge $e \in E(\mathcal{H})$.

3.1 Dynamic concentration of key variables: proof of Theorem 2

Our main goal is to understand the evolution of the colors available for each edge $e \in E(\mathcal{H})$, i.e., the size of $Q_e(i)$, where for any set of vertices $S \subseteq V(\mathcal{H})$ we more generally define

$$Q_S(i) := \big\{c \in [q] : \text{ color } c \text{ not assigned to any edge } f \in \{e_j : 1 \le j \le i\} \text{ with } f \cap S \neq \varnothing\big\}. \tag{46}$$

At the beginning of the algorithm we have $|Q_e(0)| = q$. In order to keep track of the number of available colors $|Q_e(i)|$, we need to understand changes in the colors assigned to edges adjacent to the vertices of e. To take such changes into account, for all vertices $v \in V(\mathcal{H})$ and colors $c \in [q]$ we introduce

$$Y_{v,c}(i) := \big\{f \in E(\mathcal{H}) : \ v \in f \text{ and } c \in Q_{f \setminus \{v\}}(i)\big\}, \tag{47}$$

which in case of $c \in Q_{\{v\}}(i)$ denotes the set of all edges adjacent to v that could still be colored by c (since for any $f \in Y_{v,c}(i)$ then $c \in Q_{f\setminus\{v\}}(i) \cap Q_{\{v\}}(i) = Q_f(i)$ holds). Note that initially $|Y_{v,c}(0)| = \deg_{\mathcal{H}}(v)$.

Our main technical result for the random greedy algorithm shows that, when $q \approx rm/n$ colors are used, then the above-mentioned key random variables closely follow the trajectories $|Q_e(i)| \approx \hat{q}(t)$ and $|Y_{v,c}(i)| \approx \hat{y}(t)$ during the first $m_0 \approx (1-\gamma)m$ steps, tacitly using the continuous time scaling

$$t = t(i, m) := i/m. \tag{48}$$

In particular, $\min_{e \in E(\mathcal{H})} |Q_e(m_0)| > 0$ ensures that the algorithm properly colors the first m_0 edges using at most q colors, as no edge has run out of available colors. The form of the trajectories (50)–(52) can easily be predicted via modern (pseudo-random or expected one-step changes based) heuristics, see Appendix C. Recall that we allow for repetition of edges, which is the reason why Theorem 16 includes an ad-hoc upper bound on the number m of random edges (that is significantly larger than n^r).

Theorem 16 (Dynamic concentration of the variables)**.** *For all reals $\gamma \in (0,1)$ and $\sigma, b > 0$ with*

$$b\log(1/\gamma) \le \sigma/30 \tag{49}$$

there is $n_0 = n_0(\sigma, b) > 0$ such that, for all integers $n \ge n_0$, $2 \le r \le b\log n$ and all reals $n^{1+\sigma} \le m \le n^{rn^{\sigma/4}}$, $D > 0$, the following holds for every n-vertex r-uniform hypergraph $\mathcal{H}$ satisfying the degree and codegree assumptions (2)*. With probability at least $1 - m^{-\omega(r)}$, we have $\min_{e\in E(\mathcal{H})} |Q_e(i)| > 0$ and*

$$|Q_e(i)| \;=\; \big(1 \pm \hat{e}(t)\big)\cdot \hat{q}(t) \qquad \text{for all } e \in E(\mathcal{H}), \tag{50}$$

$$|Y_{v,c}(i)| \;=\; \big(1 \pm \hat{e}(t)\big)\cdot \hat{y}(t) \qquad \text{for all } v \in V(\mathcal{H}) \text{ and } c \in [q], \tag{51}$$

for all $0 \le i \le m_0 := \lfloor(1-\gamma)m\rfloor$, where $q := \lfloor rm/n\rfloor$ and

$$\hat{q}(s) := (1-s)^r q, \qquad \hat{y}(s) := (1-s)^{r-1}D \quad \text{ and } \quad \hat{e}(s) := (1-s)^{-9r}n^{-\sigma/3}. \tag{52}$$

Remark 17. *The assumption* (49) *simply ensures $\hat{e}(t) = (1-t)^{-9r}n^{-\sigma/3} \le \gamma^{-9b\log n}\cdot n^{-\sigma/3} = n^{9b\log(1/\gamma)-\sigma/3} \le n^{-\sigma/30} = o(1)$ for all $0 \le i \le m_0$, so that estimates* (50)–(51) *imply $|Q_e(i)| \sim \hat{q}(t)$ and $|Y_{v,c}(i)| \sim \hat{y}(t)$.*

Remark 18. *The proof carries over to the case $\gamma = \gamma(n) \to 0$, provided that the assumption* (49) *is replaced by $r\log(1/\gamma)/\log n \le \sigma/30$ (to again ensure that $\hat{e}(t) \le n^{-\sigma/30} = o(1)$ holds).*

Before giving the differential equation method based proof of this result, we first show how it implies Theorem 2 by slightly increasing the number of edges from m to m', to ensure that the greedy algorithm properly colors the first $\lfloor(1-\gamma)m'\rfloor \ge m$ random edges using at most $\lfloor rm'/n\rfloor \le (1+\epsilon)rm/n$ colors.

Proof of Theorem 2. Set $\gamma := 1 - 1/(1+\delta)$, so that $b\log(1/\gamma) = b\log(1+1/\delta) \le b/\delta \le \sigma/30$ implies (49). Invoking Theorem 16 with m set to $m' := (1+\delta)m = o(n^{rn^{\sigma/4}})$ it follows that, with probability at least $1 - m^{-\omega(r)}$, the greedy algorithm properly colors the first $m_0 := \lfloor(1-\gamma)m'\rfloor = \lfloor m\rfloor = m$ random edges $e_1, \ldots, e_m$ using at most $q := \lfloor rm'/n\rfloor \le (1+\delta)rm/n$ colors, completing the proof. □

3.2 Differential equation method: proof of Theorem 16

In this subsection we prove Theorem 16 by showing $\mathbb{P}(\neg\mathcal{G}_{m_0}) \le m^{-\omega(r)}$, where $\mathcal{G}_j$ denotes the event that $\min_{e\in E(\mathcal{H})} |Q_e(i)| > 0$ and estimates (50)–(51) hold for all $0 \le i \le j$. We henceforth tacitly assume $0 \le i \le m_0$, and also that $n \ge n_0(\sigma, b)$ is sufficiently large (whenever necessary). In particular, estimate (50) implies $\min_{e\in E(\mathcal{H})} |Q_e(i)| \ge \hat{q}(t)/2 > 0$ by Remark 17. To establish (50)–(51) using the differential equation method approach to dynamic concentration [48, 3, 45], we introduce the following sequences of auxiliary random variables:

$$Q_e^{\pm}(i) \;:=\; \pm\big[|Q_e(i)| - \hat{q}(t)\big] - \hat{e}(t)\hat{q}(t) \qquad \text{for all } e \in E(\mathcal{H}), \tag{53}$$

$$Y_{v,c}^{\pm}(i) \;:=\; \pm\big[|Y_{v,c}(i)| - \hat{y}(t)\big] - \hat{e}(t)\hat{y}(t) \qquad \text{for all } v \in V(\mathcal{H}) \text{ and } c \in [q]. \tag{54}$$

Note that the desired estimates (50)–(51) follow when the four inequalities $Q_e^{\pm}(i) \le 0$ and $Y_{v,c}^{\pm}(i) \le 0$ all hold. To establish these inequalities, in Section 3.2.1 we first estimate the expected one-step changes of $|Q_e(i)|$ and $|Y_{v,c}(i)|$, which in Section 3.2.2 then enables us to show that the sequences $Q_e^{\pm}(i)$ and $Y_{v,c}^{\pm}(i)$ are supermartingales. Next, in Section 3.2.3 we bound the one-step changes of the variables, which in Section 3.2.4 then enables us to invoke a supermartingale inequality (that is optimized for the differential equation method, see Lemma 19) in order to show that $Q_e^{\pm}(i) \ge 0$ or $Y_{v,c}^{\pm}(i) \ge 0$ are extremely unlikely events.

3.2.1 Expected one-step changes

We first derive estimates for the expected one-step changes of the available colors variables $|Q_e(i)|$ and the available edges variables $|Y_{v,c}(i)|$, tacitly assuming that $0 \le i \le m_0$ and $\mathcal{G}_i$ hold. As we shall see, the expected changes (57) and (59) will be consistent with the deterministic approximations $|Q_e(i+1)| - |Q_e(i)| \approx \hat{q}(t+1/m) - \hat{q}(t) \approx \hat{q}'(t)/m = -r(1-t)^{r-1}q/m$ and $|Y_{v,c}(i+1)| - |Y_{v,c}(i)| \approx \hat{y}'(t)/m = -(r-1)(1-t)^{r-2}D/m$, which is one motivation for the choice of $\hat{q}(t)$ and $\hat{y}(t)$; see also (82)–(84) in Appendix C.

To calculate the expectation of the one-step changes $\Delta Q_e(i) := |Q_e(i+1)| - |Q_e(i)|$, we consider a color $c \in Q_e(i)$ and the event that $c \notin Q_e(i+1)$. By definition (46) of $Q_e(i)$ this only occurs if the algorithm chooses an edge f with $f \cap e \neq \varnothing$, and then assigns the color c to f. By definition (47) of $Y_{v,c}(i)$ this color assignment is only possible if $f \in \bigcup_{v \in e} Y_{v,c}(i)$, as $c \in Q_e(i) \subseteq Q_{\{v\}}(i)$ for any $v \in e$. Since the algorithm chooses both the edge $e_{i+1} \in E(\mathcal{H})$ and the color $c \in Q_{e_{i+1}}(i)$ uniformly at random, it follows that

$$\mathbb{E}(\Delta Q_e(i) \mid \mathcal{F}_i) = -\sum_{c \in Q_e(i)} \sum_{f \in \bigcup_{v \in e} Y_{v,c}(i)} \frac{1}{|E(\mathcal{H})| \cdot |Q_f(i)|}, \tag{55}$$

where $\mathcal{F}_i$ denotes, as usual, the natural filtration associated with the algorithm after i steps (which intuitively keeps track of the history algorithm, i.e., contains all the information available up to step i). Recalling the codegree assumption (2) and $r = O(\log n)$, note that the cardinality of the union $\bigcup_{v \in e} Y_{v,c}(i)$ differs from the sum $\sum_{v \in e} |Y_{v,c}(i)|$ by at most $\sum_{v \neq w \in e} \deg_{\mathcal{H}}(v,w) < n^{-\sigma/2}D < \hat{e}(t)\hat{y}(t)$. The degree assumption (2) also implies $r \cdot |E(\mathcal{H})| = \sum_{v \in V(H)} \deg_{\mathcal{H}}(v) = n \cdot (1 \pm n^{-\sigma})D$. Since the event $\mathcal{G}_i$ holds (as assumed at the beginning of Section 3.2.1), using estimates (50)–(51) it follows that

$$\begin{aligned}\mathbb{E}(\Delta Q_e(i) \mid \mathcal{F}_i) &= -\frac{\sum_{c \in Q_e(i)} \big|\bigcup_{v \in e} Y_{v,c}(i)\big|}{|E(\mathcal{H})| \cdot (1 \pm \hat{e})\hat{q}} = -\frac{\sum_{c \in Q_e(i)} \Big[\sum_{v \in e} |Y_{v,c}(i)| \pm \hat{e}\hat{y}\Big]}{|E(\mathcal{H})| \cdot (1 \pm \hat{e})\hat{q}} \\ &= -\frac{|Q_e(i)| \cdot \big[r \cdot (1 \pm \hat{e})\hat{y} \pm \hat{e}\hat{y}\big]}{|E(\mathcal{H})| \cdot (1 \pm \hat{e})\hat{q}} = -\frac{(1 \pm \hat{e})\hat{q} \cdot r \cdot (1 \pm 2\hat{e})\hat{y}}{(1 \pm n^{-\sigma})nD/r \cdot (1 \pm \hat{e})\hat{q}},\end{aligned} \tag{56}$$

where we suppressed the dependence on t to avoid clutter in the notation. Note that $1/(1+x) = 1 - x(1+o(1))$ as $x \to 0$. Since $|rm/n - q| \le 1 < n^{-\sigma} q$ and $n^{-\sigma} < \hat{e}(t) = o(1)$, using $\hat{y}(t) = (1-t)^{r-1}D$ we routinely arrive at

$$\mathbb{E}(\Delta Q_e(i) \mid \mathcal{F}_i) = -\big(1 \pm 5.5\hat{e}\big) \cdot r^2(1-t)^{r-1}/n = -\big(1 \pm 7\hat{e}\big) \cdot r(1-t)^{r-1}q/m. \tag{57}$$

To calculate the expectation of the one-step changes $\Delta Y_{v,c}(i) := |Y_{v,c}(i+1)| - |Y_{v,c}(i)|$, we consider an edge $f \in Y_{v,c}(i)$ and the event that $f \notin Y_{v,c}(i+1)$. By definition (47) of $Y_{v,c}(i)$ this only occurs if the algorithm chooses an edge e with $e \cap (f \setminus \{v\}) \neq \varnothing$, and then assigns the color c to e, which in turn is only possible if $e \in \bigcup_{w \in f \setminus \{v\}} Y_{w,c}(i)$. Proceeding similarly to (55), it follows that

$$\mathbb{E}(\Delta Y_{v,c}(i) \mid \mathcal{F}_i) = -\sum_{f \in Y_{v,c}(i)} \sum_{e \in \bigcup_{w \in f \setminus \{v\}} Y_{w,c}(i)} \frac{1}{|E(\mathcal{H})| \cdot |Q_e(i)|}, \tag{58}$$

where $|\bigcup_{w \in f \setminus \{v\}} Y_{w,c}(i)|$ differs from $\sum_{w \in f \setminus \{v\}} |Y_{w,c}(i)|$ by at most $\sum_{u \neq w \in f} \deg_{\mathcal{H}}(u,w) < \hat{e}(t)\hat{y}(t)$. Proceeding similarly to (56)–(57), using $|q - rm/n| \le 1 < n^{-\sigma} rm/n$ and $n^{-\sigma} < \hat{e}(t) = o(1)$ it follows that

$$\mathbb{E}(\Delta Y_{v,c}(i) \mid \mathcal{F}_i) = -\frac{(1 \pm \hat{e})\hat{y} \cdot (r-1) \cdot (1 \pm 2\hat{e})\hat{y}}{(1 \pm n^{-\sigma})nD/r \cdot (1 \pm \hat{e})\hat{q}} = -\big(1 \pm 7\hat{e}\big) \cdot \frac{(r-1)(1-t)^{r-2}D}{m}. \tag{59}$$

3.2.2 Supermartingale conditions

We now show that the expected one-step changes of the auxiliary variables $Q_e^{\pm}(i)$ and $Y_{v,c}^{\pm}(i)$ are negative (as required for supermartingales), tacitly assuming that $0 \le i \le m_0 - 1$ and $\mathcal{G}_i$ hold. As we shall see, the main terms in the expected changes (60) and (63) will cancel due to the estimates of Section 3.2.1, and the careful choice of $\hat{e}(t)$ then ensures that the resulting expected changes (62) and (64) are indeed negative (by ensuring that the ratios $e'_X(t)/e_X(t)$ of the below-defined error functions $e_X(t)$ are sufficiently large).

For the one-step changes $\Delta Q_e^{\pm}(i) := Q_e^{\pm}(i+1) - Q_e^{\pm}(i)$, set $e_Q(s) := \hat{e}(s)\hat{q}(s) = (1-s)^{-8r} n^{-\sigma/3} q$. Recalling $t = i/m$, by applying Taylor's theorem with remainder it follows that

$$\begin{aligned}\mathbb{E}(\Delta Q_e^{\pm}(i) \mid \mathcal{F}_i) &= \pm\Big[\mathbb{E}(\Delta Q_e(i) \mid \mathcal{F}_i) - \Big[\hat{q}\big(t + 1/m\big) - \hat{q}(t)\Big]\Big] - \Big[e_Q\big(t + 1/m\big) - e_Q(t)\Big] \\ &= \pm\Big[\mathbb{E}(\Delta Q_e(i) \mid \mathcal{F}_i) - \frac{\hat{q}'(t)}{m}\Big] - \frac{e'_Q(t)}{m} + O\Big(\max_{s \in [0, m_0/m]} \frac{|\hat{q}''(s)| + |e''_Q(s)|}{m^2}\Big).\end{aligned} \tag{60}$$

The key point is that the derivative $\hat{q}'(t)/m = -r(1-t)^{r-1}q/m$ equals the main term in (57), and that the other term in (57) satisfies $7\hat{e}(t) \cdot r(1-t)^{r-1}q/m = 7r(1-t)^{-1}e_Q(t)/m$. Furthermore, using the estimate from Remark 17 together with $m \geq n^{1+\sigma}$ and $r = O(\log n)$, for all $s \in [0, m_0/m]$ it is routine to see that

$$\frac{|\hat{q}''(s)| + |e_Q''(s)|}{m} = O\biggl(\frac{r^2 q + r^2(1-s)^{-8r-2}n^{-\sigma/3}q}{m}\biggr) = o(n^{-\sigma/3}q). \tag{61}$$

Putting things together, now the crux is that $e_Q'(t) = 8r(1-t)^{-1}e_Q(t) = \Omega(n^{-\sigma/3}q)$ implies

$$\mathbb{E}(\Delta Q_e^{\pm}(i) \mid \mathcal{F}_i) \leq \frac{7r(1-t)^{-1}e_Q(t) - e_Q'(t) + o(n^{-\sigma/3}q)}{m} < 0. \tag{62}$$

For the one-step changes $\Delta Y_{v,c}^{\pm}(i) := Y_{v,c}^{\pm}(i+1) - Y_{v,c}^{\pm}(i)$, set $e_Y(s) := \hat{e}(s)\hat{y}(s) = (1-s)^{-8r-1}n^{-\sigma/3}D$. Proceeding similarly to (60), we obtain

$$\mathbb{E}(\Delta Y_{v,c}^{\pm}(i) \mid \mathcal{F}_i) = \pm\biggl[\mathbb{E}(\Delta Y_{v,c}(i) \mid \mathcal{F}_i) - \frac{\hat{y}'(t)}{m}\biggr] - \frac{e_Y'(t)}{m} + O\biggl(\max_{s\in[0,m_0/m]} \frac{|\hat{y}''(s)| + |e_Y''(s)|}{m^2}\biggr). \tag{63}$$

The key point is that the derivative $\hat{y}'(t)/m = -(r-1)(1-t)^{r-2}D/m$ equals the main term in (59), and that the other term in (59) satisfies $7\hat{e}(t) \cdot (r-1)(1-t)^{r-2}D/m = 7(r-1)(1-t)^{-1}e_Y(t)/m$. Analogously to (61), it is routine to see that $|\hat{y}''(s)| + |e_Y''(s)| = o(n^{-\sigma/3}Dm)$ for all $s \in [0, m_0/m]$. Putting things together similarly to (62), here the crux is that $e_Y'(t) = (8r+1)(1-t)^{-1}e_Y(t) = \Omega(n^{-\sigma/3}D)$ implies

$$\mathbb{E}(\Delta Y_{v,c}^{\pm}(i) \mid \mathcal{F}_i) \leq \frac{7(r-1)(1-t)^{-1}e_Y(t) - e_Y'(t) + o(n^{-\sigma/3}D)}{m} < 0. \tag{64}$$

3.2.3 Bounds on one-step changes

We next derive bounds on the one-step changes of the variables $|Q_e(i)|$ and $|Y_{v,c}(i)|$ (as required by the supermartingale inequality in Section 3.2.4), tacitly assuming that $0 \leq i \leq m_0$ and $\mathcal{G}_i$ hold. As we shall see, the expected changes (66) and (68) are easy to bound due to step-wise monotonicity of the variables.

The one-step changes $\Delta Q_e(i) = |Q_e(i+1)| - |Q_e(i)|$ of the available colors satisfy

$$|\Delta Q_e(i)| \leq 1. \tag{65}$$

Since $|Q_e(i)|$ is step-wise decreasing, by inserting $\hat{e}(t) = o(1)$ and $r(1-t)^{r-1} \leq r$ into (57) we obtain

$$\mathbb{E}(|\Delta Q_e(i)| \mid \mathcal{F}_i) = -\,\mathbb{E}(\Delta Q_e(i) \mid \mathcal{F}_i) \leq 2rq/m. \tag{66}$$

The one-step changes $\Delta Y_{v,c}(i) = |Y_{v,c}(i+1)| - |Y_{v,c}(i)|$ of the available edges satisfy

$$|\Delta Y_{v,c}(i)| \leq \sum_{w \in e_{i+1}\setminus\{v\}} \deg_{\mathcal{H}}(v,w) \leq r \cdot n^{-\sigma}D \tag{67}$$

due to the codegree assumption (2). Since $|\Delta Y_{v,c}(i)|$ is step-wise decreasing, by inserting $\hat{e}(t) = o(1)$ and $(r-1)(1-t)^{r-2} \leq r$ into (59) we also obtain

$$\mathbb{E}(|\Delta Y_{v,c}(i)| \mid \mathcal{F}_i) = -\,\mathbb{E}(\Delta Y_{v,c}(i) \mid \mathcal{F}_i) \leq 2rD/m. \tag{68}$$

3.2.4 Supermartingale estimates

We finally bound $\mathbb{P}(\neg\mathcal{G}_{m_0})$ by focusing on the first step where the estimates (50)–(51) are violated, which by the discussion below (53)–(54) can only happen if $Q_e^{\pm}(i) \leq 0$ or $Y_{v,c}^{\pm}(i) \leq 0$ is violated. Our main tool for bounding the probabilities of these 'first bad events' will be the following Freedman type supermartingale inequality: it is optimized for the differential equation method approach to dynamic concentration, where supermartingales S_i are constructed by adding a deterministic quantity to a random variable X_i, cf. the definition of $Q_e^{\pm}(i)$ and $Y_{v,c}^{\pm}(i)$ in (53)–(54). Here the convenient point is that Lemma 19 only assumes upper bounds on the one-step changes of X_i (and not of S_i, as usual, cf. [4, Lemma 3.4]).

Lemma 19. *Let $(S_i)_{i\geq 0}$ be a supermartingale adapted to the filtration $(\mathcal{F}_i)_{i\geq 0}$. Assume that $S_i = X_i + D_i$, where X_i is $\mathcal{F}_i$-measurable and D_i is $\mathcal{F}_{\max\{i-1,0\}}$-measurable. Writing $\Delta X_i := X_{i+1} - X_i$, assume that $\max_{i\geq 0} |\Delta X_i| \leq C$ and $\sum_{i\geq 0} \mathbb{E}(|\Delta X_i| \mid \mathcal{F}_i) \leq V$. Then, for all $z > 0$,*

$$\mathbb{P}\big(S_i \geq S_0 + z \text{ for some } i \geq 0\big) \;\leq\; \exp\bigg(-\frac{z^2}{2C(V+z)}\bigg). \tag{69}$$

Proof. Writing $\Delta S_i := S_{i+1} - S_i$, set $M_i := S_i - \sum_{0\leq j<i} \mathbb{E}(\Delta S_j \mid \mathcal{F}_j)$. Note that $S_i = X_i + D_i$ implies

$$\Delta M_i := M_{i+1} - M_i = \Delta S_i - \mathbb{E}(\Delta S_i \mid \mathcal{F}_i) = \Delta X_i - \mathbb{E}(\Delta X_i \mid \mathcal{F}_i),$$

which readily gives $\mathbb{E}(\Delta M_i \mid \mathcal{F}_i) = 0$ and $\max_{i\geq 0} |\Delta M_i| \leq 2 \cdot C$. Note that we also have

$$\mathrm{Var}(\Delta M_i \mid \mathcal{F}_i) = \mathrm{Var}(\Delta X_i \mid \mathcal{F}_i) \leq \mathbb{E}(\Delta X_i^2 \mid \mathcal{F}_i) \leq C \cdot \mathbb{E}(|\Delta X_i| \mid \mathcal{F}_i), \tag{70}$$

so that $\sum_{i\geq 0} \mathrm{Var}(\Delta M_i \mid \mathcal{F}_i) \leq C \cdot V$. Clearly $M_0 = S_0$. Also $M_i \geq S_i$, since $(S_i)_{i\geq 0}$ is a supermartingale. Hence a standard application of Freedman's martingale inequality (see [15] or [43, Lemma 2.2]) yields

$$\mathbb{P}\big(S_i \geq S_0 + z \text{ for some } i \geq 0\big) \leq \mathbb{P}\big(M_i \geq M_0 + z \text{ for some } i \geq 0\big) \leq \exp\bigg(-\frac{z^2}{2(CV + 2C \cdot z/3)}\bigg), \tag{71}$$

which completes the proof of inequality (69). $\square$

Turning to the details, we define the stopping time I as the minimum of m_0 and the first step $i \geq 0$ where $\mathcal{G}_i$ fails. Writing $i \wedge I := \min\{i, I\}$, as usual, by our above discussion it follows that

$$\begin{aligned}\mathbb{P}\big(\neg\mathcal{G}_{m_0}\big) \leq &\sum_{e\in E(\mathcal{H})} \sum_{\tau\in\{+,-\}} \mathbb{P}\big(Q_e^\tau(i\wedge I) \geq 0 \text{ for some } i \geq 0\big)\\ &+ \sum_{v\in V(\mathcal{H})} \sum_{c\in[q]} \sum_{\tau\in\{+,-\}} \mathbb{P}\big(Y_{v,c}^\tau(i\wedge I) \geq 0 \text{ for some } i \geq 0\big).\end{aligned} \tag{72}$$

Note that initially $|Q_e(i)| = q$ and $|Y_{v,c}(0)| = \deg_{\mathcal{H}}(v)$, which in view of the degree assumption (2) and the definitions (53)–(54) of $Q_e^\tau(0)$ and $Y_{v,c}^\tau(0)$ gives the initial value estimates

$$\begin{aligned}Q_e^\tau(0\wedge I) = Q_e^\tau(0) &= -\hat{e}(0)q = -n^{-\sigma/3}q,\\ Y_{v,c}^\tau(0\wedge I) = Y_{v,c}^\tau(0) &= O(n^{-\sigma}D) - \hat{e}(0)D \leq -n^{-\sigma/3}D/2.\end{aligned}$$

Noting that the estimates from Sections 3.2.2–3.2.3 apply for $0 \leq i \leq I-1$ (since then $0 \leq i \leq m_0 - 1$ and $\mathcal{G}_i$ hold), the point is that the stopped sequence $S_i := Q_e^\tau(i\wedge I)$ is a supermartingale with $S_0 = -n^{-\sigma/3}q$, to which Lemma 19 can be applied with $X_i = \tau|Q_e(i\wedge I)|$, $C = 1$ and $V = m_0 \cdot 2rq/m = O(rq)$. Invoking inequality (69) with $z = n^{-\sigma/3}q$, using $q = \Omega(rn^\sigma)$ together with $m^r \leq n^{r^2 n^{\sigma/4}}$ and $r = O(\log n)$ it follows that

$$\mathbb{P}\big(Q_e^\tau(i\wedge I) \geq 0 \text{ for some } i \geq 0\big) \leq \exp\Big\{-\Theta(n^{-2\sigma/3}q/r)\Big\} \leq \exp\Big\{-\Theta(n^{\sigma/3})\Big\} \leq m^{-\omega(r)}. \tag{73}$$

Similarly, the sequence $S_i := Y_{v,c}^\tau(i\wedge I)$ is a supermartingale with $S_0 \leq -n^{-\sigma/3}D/2$, to which Lemma 19 can be applied with $X_i = \tau|Y_{v,c}(i\wedge I)|$, $C = rn^{-\sigma}D$ and $V = m_0 \cdot 2rD/m = O(rD)$. Invoking inequality (69) with $z = n^{-\sigma/3}D/2$, it follows analogously to (73) that

$$\mathbb{P}\big(Y_{v,c}^\tau(i\wedge I) \geq 0 \text{ for some } i \geq 0\big) \leq \exp\Big\{-\Theta(n^{\sigma/3}/r^2)\Big\} \leq m^{-\omega(r)}. \tag{74}$$

Inserting (73)–(74) into inequality (72), noting $|V(H)| = n \leq m$, $|E(\mathcal{H})| \leq n^r \leq m^r$ and $q \leq m$ it then follows that $\mathbb{P}(\neg\mathcal{G}_{m_0}) \leq m^{-\omega(r)}$, which completes the proof of Theorem 16. $\square$

4 Concluding remarks

The main remaining open problem is to determine the typical asymptotic behavior of the Prague dimension $\dim_{\mathrm{P}}(G_{n,p}) \approx \mathrm{cc}'(G_{n,1-p})$ as well as the clique covering and partition numbers $\mathrm{cc}(G_{n,p})$ and $\mathrm{cp}(G_{n,p})$, i.e., to refine the estimates from Theorems 1, 4 and 5. Here edge-probability $p = 1/2$ is of special interest, since this would reveal the asymptotics of these intriguing parameters for almost all n-vertex graphs.

Problem 1. *Determine the whp asymptotics of the parameters $cc(G_{n,p})$, $cp(G_{n,p})$, $cc_{\Delta}(G_{n,p})$, and $cc'(G_{n,p})$ for constant edge-probabilities $p \in (0,1)$.*

4.1 Non-trivial lower bounds for dense random graphs

For constant edge-probabilities $p \in (0,1)$ our understanding of the asymptotics remains unsatisfactory, even on a heuristic level. Indeed, it is well-known that the largest clique of $G_{n,p}$ whp has size $s \sim 2\log_{1/p} n$, which together with the simple lower bound reasoning for Theorem 4 makes it tempting to speculate that perhaps $\mathrm{cc}(G_{n,p}) \sim \binom{n}{2}p/\binom{s}{2}$ holds whp. However, Lemma 20 shows that this natural guess is false, by further improving the simple lower bound (which for $p = 1/2$ was already noted in [5]). The analogous speculation $\mathrm{cc}_{\Delta}(G_{n,p}) \sim np/(s-1)$ is also refuted by Lemma 20, whose proof we defer to Appendix A.

Lemma 20. *If $p = p(n)$ satisfies $n^{-o(1)} \le p \le 1 - n^{-o(1)}$, then for any $\epsilon \in (0,1)$ whp*

$$cc(G_{n,p}) \ \ge\ (1-\epsilon)\cdot\bigl(1+\varphi(p)\bigr)\tbinom{n}{2}p/\tbinom{s}{2}, \tag{75}$$

$$cc_{\Delta}(G_{n,p}) \ \ge\ (1-\epsilon)\cdot\bigl(1+\varphi(p)\bigr)np/(s-1), \tag{76}$$

where $s := \lceil 2\log_{1/p} n\rceil$ and $\varphi(p) := (1-p)\log(1-p)/(p\log p)$. The function $\varphi : (0,1) \to (0,\infty)$ is increasing, with $\lim_{p\searrow 0}\varphi(p) = 0$, $\varphi(1/2) = 1$, and $\lim_{p\nearrow 1}\varphi(p) = \infty$.

For Problem 1 the main conceptual message of Lemma 20 is as follows: it simply is not enough to mainly use cliques of near maximal size, which in turn indicates that the correct asymptotics are somewhat tricky.

4.2 Asymptotics for sparse random graphs

We now record strengthenings of Theorems 4–5 for many small edge-probabilities $p = p(n) \to 0$, where the asymptotics follow from Pippenger–Spencer type hypergraph results. As we shall see, here the crux is that when all cliques have size $O(1)$, then it suffices to simply cover a $1 - o(1)$ fraction of the relevant edges.

Theorem 21. *If $p = p(n)$ satisfies $n^{-2/(s+1)} \ll p \ll n^{-2/(s+2)}$ for some fixed integer $s \ge 3$, then $cc(G_{n,p})$ and $cp(G_{n,p})$ are whp both asymptotic to $\binom{n}{2}p/\binom{s}{2}$.*

We leave it as an open problem to determine the whp asymptotics for $p = \Theta(n^{-2/(s+1)})$, and now outline the proof of Theorem 21, which uses $\mathrm{cc}(G_{n,p}) \le \mathrm{cp}(G_{n,p})$. The lower bound on $\mathrm{cc}(G_{n,p})$ is routine: the expected number of edges in cliques of size at least $s+1$ is at most $\sum_{k\ge s+1}\binom{k}{2}\binom{n}{k}p^{\binom{k}{2}} \ll \binom{n}{2}p$, which makes it easy to see that whp $\mathrm{cc}(G_{n,p}) \ge (1-o(1))\binom{n}{2}p/\binom{s}{2}$. For the upper bound on $\mathrm{cp}(G_{n,p})$ we shall mimic the natural strategy of Kahn and Park [23] for $s = 3$: using Kahn's fractional version of Pippenger's hypergraph packing result [23, Theorem 7.1] it is not difficult[7] to see that $G_{n,p}$ whp contains a collection $\mathcal{C}$ of $|\mathcal{C}| = (1-o(1))\binom{n}{2}p/\binom{s}{2}$ edge-disjoint cliques K_s. Writing $\mathcal{U}$ for the edges of $G_{n,p}$ not covered by the cliques in $\mathcal{C}$, it then easily follows that whp $\mathrm{cp}(G_{n,p}) \le |\mathcal{C}| + |\mathcal{U}| \le (1+o(1))\binom{n}{2}p/\binom{s}{2}$, as desired.

Theorem 22. *If $p = p(n)$ satisfies $(\log n)^{\omega(1)}n^{-2/(s+1)} \le p \ll n^{-2/(s+2)}$ for some fixed integer $s \ge 3$, then $cc_{\Delta}(G_{n,p})$ and $cc'(G_{n,p})$ are whp both asymptotic to $np/(s-1)$.*

Remark 23. *These asymptotics remain valid when the definitions of $cc_{\Delta}(G_{n,p})$ and $cc'(G_{n,p})$ are restricted to clique partitions of the edges (instead of clique coverings).*

[7] We consider the auxiliary hypergraph $\mathcal{H}$, where the vertices correspond to the edges of $G_{n,p}$ and the edges correspond to the edge-sets of the cliques K_s of $G_{n,p}$. The technical conditions of [23, Theorem 7.1] required for mimicking [23, Section 7] can then be verified using (careful applications of) standard tail bounds such as Lemma 11 and [46, Theorems 30 and 32].

We leave it as an open problem to determine the whp asymptotics for $p = (\log n)^{O(1)} n^{-2/(s+1)}$, and now outline the proof of Theorem 22, which uses $\mathrm{cc}_\Delta(G_{n,p}) \le \mathrm{cc}'(G_{n,p})$. The lower bound on $\mathrm{cc}_\Delta(G_{n,p})$ is routine: the expected number of edges in cliques of size at least $s+1$ containing a fixed vertex v is at most $\sum_{k \ge s+1} \binom{k}{2}\binom{n-1}{k-1} p^{\binom{k}{2}} \ll np$, which makes it easy to see that whp $\mathrm{cc}_\Delta(G_{n,p}) \ge (1-o(1))np/(s-1)$. Turning to the upper bound on $\mathrm{cc}'(G_{n,p})$, using a pseudo-random variant of Pippenger's packing result due to Ehard, Glock and Joos [11], it is not difficult[8] to see that $G_{n,p}$ whp contains a collection $\mathcal{C}$ of edge-disjoint cliques K_s where each vertex is contained in $(1-o(1))np/(s-1)$ cliques of $\mathcal{C}$. Writing $\mathcal{U}$ for the edges of $G_{n,p}$ not covered by the cliques in $\mathcal{C}$, using Pippenger and Spencer's chromatic index result [37] and Vizing's theorem it then is not difficult to see that whp $\mathrm{cc}'(G_{n,p}) \le \chi'(\mathcal{C}) + \chi'(\mathcal{U}) \le (1+o(1))np/(s-1)$, as desired.

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A Lower bounds: proof of Lemma 20

Proof of Lemma 20. Writing $\mathcal{S}$ for the event that the largest clique of $G_{n,p}$ has size at most $s = \lceil 2\log_{1/p} n\rceil$, it well-known that $\mathcal{S}$ holds whp (by a straightforward first moment argument). Writing $\mathcal{E}$ for the event that $G_{n,p}$ contains $(1\pm\epsilon)\binom{n}{2}p$ edges for $\epsilon := n^{-1/2}$, say, it is easy to see that $\mathcal{E}$ holds whp (using Chebychev's inequality). Furthermore, recalling $\varphi(p) = (1-p)\log(1-p)/(p\log p)$, the probability that $G_{n,p}$ equals any fixed spanning subgraph $G \subseteq K_n$ with $e(G) = (1\pm\epsilon)\binom{n}{2}p$ edges is routinely seen to be at most

$$\begin{aligned}\Pi &:= \max_{m\in(1\pm\epsilon)\binom{n}{2}p} p^m(1-p)^{\binom{n}{2}-m}\\ &= \max_{m\in(1\pm\epsilon)\binom{n}{2}p} \exp\Bigg(-\tbinom{n}{2}p\bigg[\frac{m}{\binom{n}{2}p}+\bigg(1-\frac{m}{\binom{n}{2}}\bigg)\frac{\log(1-p)}{p\log p}\bigg]\cdot\log(1/p)\Bigg)\\ &\le \exp\Bigg(-\tbinom{n}{2}p\bigg[1-\epsilon+\bigg(1-\frac{\epsilon p}{1-p}\bigg)\varphi(p)\bigg]\cdot\log(1/p)\Bigg)\\ &\le \exp\Big(-(1-o(1))\cdot\tbinom{n}{2}p\big(1+\varphi(p)\big)\cdot\log(1/p)\Big),\end{aligned} \tag{77}$$

where we used $\varphi(p)\ge 0$ as well as $\epsilon = o(1)$ and $\epsilon p/(1-p) = o(1)$ for the last inequality.

For the clique covering number $\mathrm{cc}(G_{n,p})$, the crux is that there are at most

$$\binom{n+s}{s}^T \le o(n^{sT})$$

many collections $\{C_1, \ldots, C_t\}$ with $t \le T$ that are a clique covering for some graph $G \subseteq K_n$ with largest clique of size at most s. Hence, since each clique covering uniquely determines the entire edge-set and thus the underlying spanning subgraph $G \subseteq K_n$, it follows by a union bound argument that

$$\mathbb{P}(\mathrm{cc}(G_{n,p}) \le T) \ \le\ \mathbb{P}(\neg\mathcal{S} \text{ or } \neg\mathcal{E}) \ +\ o(n^{sT}) \cdot \Pi. \tag{78}$$

Note that $\mathbb{P}(\neg\mathcal{S} \text{ or } \neg\mathcal{E}) = o(1)$ and $s \log n \sim \binom{s}{2} \cdot \log(1/p)$. In view of inequality (77), for any $\epsilon \in (0,1)$ it follows that (78) is at most $o(1)$ when $T \le (1-\epsilon) \cdot (1+\varphi(p))\binom{n}{2}p/\binom{s}{2}$, establishing (75).

Turning to the thickness $\mathrm{cc}_\Delta(G_{n,p})$, we associate each clique covering $\mathcal{C}$ of some graph $G \subseteq K_n$ with an auxiliary bipartite graph $\mathcal{B}$ on vertex-set $[n] \cup \mathcal{C}$, where $v \in [n]$ and $C_i \in \mathcal{C}$ are connected by an edge whenever $v \in V(C_i)$. If the thickness of $\mathcal{C}$ is at most T, then in $\mathcal{B}$ the degree of each $v \in [n]$ is at most $\lfloor T \rfloor$, which also gives $|\mathcal{C}| \le n\lfloor T \rfloor$. Since the structure of the auxiliary bipartite graph $\mathcal{B}$ uniquely determines $\mathcal{C}$ (as the neighbors of C_i in $\mathcal{B}$ determine the clique vertex-set $V(C_i)$), it follows that there are at most

$$\binom{n\lfloor T \rfloor + \lfloor T \rfloor}{\lfloor T \rfloor}^n \ \le\ O\big((6n)^{nT}\big)$$

many collections $\mathcal{C}$ with thickness at most T that are a clique covering of some graph $G \subseteq K_n$. Since each such $\mathcal{C}$ uniquely determines the underlying spanning subgraph $G \subseteq K_n$, we obtain similarly to (78) that

$$\mathbb{P}(\mathrm{cc}_\Delta(G_{n,p}) \le T) \ \le\ \mathbb{P}(\neg\mathcal{E}) \ +\ O\big((6n)^{nT}\big) \cdot \Pi. \tag{79}$$

Note that $\mathbb{P}(\neg\mathcal{E}) = o(1)$ and $n \log(6n) \sim \binom{n}{2} \log(1/p) \cdot (s-1)/n$. In view of inequality (77), for any $\epsilon \in (0,1)$ it follows that (79) is at most $o(1)$ when $T \le (1-\epsilon) \cdot (1+\varphi(p))np/(s-1)$, completing the proof of (76). □

B Variant of Theorem 2: proof of Corollary 10

Proof of Corollary 10. Choosing $\xi = \xi(\delta) \in (0, 1/16]$ such that $(1+\delta)(1+\xi)/(1-4\xi)^2 \le 1+2\delta$, set $m_0 := \lfloor (1+\xi)m \rfloor$, $m_1 := \lfloor m_0/(1-4\xi)^2 \rfloor$, and $c := (1+\delta)rm_1/n$. Let $\mathcal{H}_i^*$ be chosen uniformly at random from all $\binom{|E(\mathcal{H})|}{i}$ subhypergraphs of $\mathcal{H}$ with exactly i edges. Since $\mathcal{H}_q$ conditioned on having exactly i edges has the same distribution as $\mathcal{H}_i^*$, by the law of total probability and monotonicity it follows that

$$\begin{aligned} \mathbb{P}(\chi'(\mathcal{H}_q) \ge c) \ &\le\ \mathbb{P}(|E(\mathcal{H}_q)| > m_0) + \sum_{0 \le i \le m_0} \mathbb{P}(\chi'(\mathcal{H}_i^*) \ge c)\, \mathbb{P}(|E(\mathcal{H}_q)| = i) \\ &\le\ n^{-\omega(r)} + \mathbb{P}(\chi'(\mathcal{H}_{m_0}^*) \ge c), \end{aligned} \tag{80}$$

where we used standard Chernoff bounds (such as [21, Theorem 2.1]) and $\mathbb{E}|E(\mathcal{H}_q)| = |E(\mathcal{H})|q = m \ge n^{1+\sigma} \gg r \log n$. Sequentially choosing the random edges $e_1, \ldots, e_{m_1} \in E(\mathcal{H})$ of $\mathcal{H}_{m_1}$ as defined in Theorem 2, note that $e_{i+1} \in E(\mathcal{H}) \setminus \{e_1, \ldots, e_i\}$ holds with probability at least $1 - m_1/e(\mathcal{H}) > 1 - 4\xi$, as $m_1 < 4m \le 4\xi e(\mathcal{H})$. Since we can equivalently construct the edge-set $\{f_1, \ldots, f_{m_0}\}$ of $\mathcal{H}_{m_0}^*$ by sequentially choosing $f_{i+1} \in E(\mathcal{H}) \setminus \{f_1, \ldots, f_i\}$ uniformly at random, a natural coupling of $\mathcal{H}_{m_1}$ and $\mathcal{H}_{m_0}^*$ thus satisfies

$$\mathbb{P}(\mathcal{H}_{m_0}^* \subseteq \mathcal{H}_{m_1}) \ \ge\ \mathbb{P}(\mathrm{Bin}(m_1, 1-4\xi) \ge m_0) \ \ge\ 1 - n^{-\omega(r)},$$

where we used standard Chernoff bounds and that $m_1(1-4\xi) > m_0/(1-\xi)$ for $n \ge n_0(\xi)$. Hence

$$\mathbb{P}(\chi'(\mathcal{H}_{m_0}^*) \ge c) \ \le\ \mathbb{P}(\chi'(\mathcal{H}_{m_1}) \ge c) + n^{-\omega(r)} \ \le\ n^{-\omega(r)}, \tag{81}$$

where we invoked Theorem 2 with m set to m_1 (which applies since $n^{1+\sigma} \le m \le m_1 < 4\xi e(\mathcal{H}) < n^r$). This completes the proof by combining (80) and (81) with $c \le (1+2\delta)rm/n$. □

C Heuristics: random greedy edge coloring algorithm

In this appendix we give, for the greedy coloring algorithm from Section 3, two heuristic explanations for the trajectories $|Q_e(i)| \approx \hat{q}(t)$ and $|Y_{v,c}(i)| \approx \hat{y}(t)$ that these random variables follow, where $t = t(i,m) = i/m$.

For our first *pseudo-random heuristic*, we write $E_i = \{e_1, \ldots, e_i\}$ for the multi-set of edges appearing during the first i steps of the algorithm. Ignoring that edges can appear multiple times, our pseudo-random ansatz is that the edges in E_i and their assigned colors are approximately independent with

$$\mathbb{P}\big(e \text{ in } E_i \text{ and colored } c\big) \approx \frac{|E_i|}{|E(\mathcal{H})|} \cdot \frac{1}{q} \approx \frac{i}{nD/r} \cdot \frac{1}{rm/n} = \frac{t}{D} =: p(t, D) = p,$$

where independence only holds with respect to colorings that are proper, i.e., possible in the algorithm. Using this heuristic ansatz, we now consider the event $\mathcal{E}_{v,c}$ that no edge $f \in E_i$ with $v \in f$ is colored c. Exploiting that no two distinct edges containing v can receive the same color in the algorithm (since this coloring would not be proper), our pseudo-random ansatz and the degree assumption (2) then suggests that

$$\mathbb{P}(\neg\mathcal{E}_{v,c}) = \sum_{f \in E(\mathcal{H}): v \in f} \mathbb{P}(f \text{ in } E_i \text{ and colored } c) \approx D \cdot p = t.$$

Since for every pair of vertices there are only at most $n^{-\sigma} D$ edges containing both (by the codegree assumption), for $\ell = o(\log n)$ distinct vertices $v_1, \ldots, v_\ell$ our pseudo-random ansatz also loosely suggests that

$$\mathbb{P}\Big(\bigcap_{i \in [\ell]} \mathcal{E}_{v_i,c}\Big) \approx \prod_{i \in [\ell]} \mathbb{P}(\mathcal{E}_{v_i,c}) + O\big(\ell^2 \cdot n^{-\sigma} D \cdot p\big) \approx (1-t)^\ell.$$

Recalling (46) from Section 3, using linearity of expectation we then anticipate $|Q_e(i)| \approx \hat{q}(t)$ based on

$$\mathbb{E}\,|Q_e(i)| = \sum_{c \in [q]} \mathbb{P}\big(c \in Q_e(i)\big) = \sum_{c \in [q]} \mathbb{P}\Big(\bigcap_{v \in e} \mathcal{E}_{v,c}\Big) \approx q \cdot (1-t)^r = \hat{q}(t).$$

Mimicking this reasoning, recalling (47) we similarly anticipate $|Y_{v,c}(i)| \approx \hat{y}(t)$ based on

$$\mathbb{E}\,|Y_{v,c}(i)| = \sum_{f \in E(\mathcal{H}): v \in f} \mathbb{P}\big(c \in Q_{f \setminus \{v\}}(i)\big) \approx D \cdot (1-t)^{r-1} = \hat{y}(t).$$

In our second *expected one-step changes heuristic* we assume for simplicity that there are deterministic approximations $|Q_e(i)| \approx f(t)q$ and $|Y_{v,c}(i)| \approx g(t)D$. Using these approximations and $q \approx rm/n$, the calculations leading to (55)–(56) and (58)–(59) in Section 3.2.1 then suggest that

$$\mathbb{E}\big(|Q_e(i+1)| - |Q_e(i)| \;\big|\; \mathcal{F}_i\big) \approx -\frac{f(t)q \cdot r \cdot g(t)D}{nD/r \cdot f(t)q} \approx -\frac{rg(t)q}{m}, \tag{82}$$

$$\mathbb{E}\big(|Y_{v,c}(i+1)| - |Y_{v,c}(i)| \;\big|\; \mathcal{F}_i\big) \approx -\frac{g(t)D \cdot (r-1) \cdot g(t)D}{nD/r \cdot f(t)q} \approx -\frac{(r-1)g^2(t)D}{f(t)m}, \tag{83}$$

where $\mathcal{F}_i$ denotes the natural filtration of the algorithm after i steps. Since the left-hand sides of (82)–(83) are approximately equal to $[f(t+1/m) - f(t)]q \approx f'(t)q/m$ and $g'(t)D/m$, respectively, we anticipate

$$f'(t) = -rg(t) \quad \text{ and } \quad g'(t) = -(r-1)g^2(t)/f(t). \tag{84}$$

Noting $|Q_e(0)| = q$ and $|Y_{v,c}(0)| \approx D$, we also anticipate $f(0) = g(0) = 1$. The solutions $f(t) = (1-t)^r$ and $g(t) = (1-t)^{r-1}$ then make $|Q_e(i)| \approx f(t)q = \hat{q}(t)$ and $|Y_{v,c}(i)| \approx g(t)D = \hat{y}(t)$ plausible.